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FOREWORD

IRE Standards on Symbols and Terminology for Feedback Control Systems

Although the standards on symbols and terminology were published in the November, 1955, and January, 1956, issues of the Proceedings, they are reproduced in this Transactions to remind control engineers that standards do exist. Standard symbols and terminology definitely increase the efficiency of obtaining information from technical material. Unfortunately, all papers in this issue of the Transactions do not use the standard symbols, but an attempt will be made in the future to have authors use standards as much as possible.

Considerable debate, argument and compromise was necessary to arrive at these standards; each term was carefully chosen. Most of the terminology is compatible with other proposed terminology in the field. However, it is intended to be more fundamental, applicable to a general feedback loop as well as for a specific form. For example, the loop input signal may be applied anywhere in the loop; the loop output signal may be chosen anywhere in the loop.

It was felt that this was necessary to define adequately the characteristics of the teterogenous control loops, network and systems which are encountered by the diversified interests of the IRE professional groups. Any criticism of these terms or suggestions for modification will be seriously considered by Subcommittee 26.1 and Committee 26.0, and it will be discussed in the Transactions.

Meanwhile, careful study and use of these standard symbols and terminology by members of the PGAC will simplify the problem of conveying new ideas with a minimum of effort.

Abstracts of Minutes of the Meeting of the Administrative Committee

The committee felt that PGAC members might desire to know the problems, interests and scope of the group as a whole. The minutes of the Administrative Committee meetings are to be abstracted and included in the Transactions for this purpose. Anyone interested in serving on the Administrative Committee or national subcommittees should write to the PGAC chairman or the editor of the Transactions. Assistance on future committee work will be needed.

Index of PGAC Membership

The membership of the PGAC has increased steadily since its inception. To illustrate its growth and to indicate the membership in various sections, the names of all the members are given in this issue. This should be an aid to those wishing to form chapters which will strengthen the PGAC and enrich the Transactions with papers from local meetings.

It is hoped that more IRE members interested in control work will join the PGAC and identify themselves professionally in this automatic control index. The index will be published periodically in the future.

THEORY, PRACTICE OR INSTRUCTION?

The first four papers in this issue are essentially mathematical in nature. Although examples of applying the theories are given, there are no actual measurements or applications to physical equipment correlating the theory with practice. In general, the ideas should be contributions to the field of automatic control, and it is hoped that they may be useful in solving practical problems or in stimulating other unique methods of analysis or synthesis. Papers on the application of these and other theories to control systems will be particularly welcome.

It is interesting to note that responses of the nonlinear systems are given exactly, without recourse to the popular describing function or phase plane methods. In fact, one nonlinear system is synthesized in such a manner that it becomes a relatively simple system which is easy to analyze exactly. Perhaps this idea could be applied to a larger class of systems. As pointed out in one paper, exact solutions can be used to check the current approximation techniques and others which may be developed.

The last two papers in this issue are tutorial. Relatively short tutorial papers are basically difficult to write. From the reader's viewpoint the background material may be too extensive or too brief, the subject matter may be too sketchy or unbalanced, some pertinent facts may be omitted. All of the field cannot be covered, and few concrete examples can be given. The last paper on the survey of the sampled-data techniques contains conclusions about methods of analysis which undoubtedly would be modified by some readers more familiar with other techniques. Comments on these tutorial papers and suggestions for other topics will be welcome. Actually, reactions to the first PGAC Transactions were favorable, but few. More problems, discussions, notes, opinions and comments on improving future issues would be desirable. In particular, it would be interesting to have opinions as to whether papers on theory, practice or instruction should be predominant.

Of course the Transactions are issued for those interested in automatic control and in advancing knowledge in the field. This can be done most effectively by using the Transactions as a medium to interchange ideas, stimulate discussions and define the issues which confront the automatic control engineer.

THE ISSUE IN BRIEF

FINAL VALUE CONTROLLER SYNTHESIS .... M. V. Mathews and C. W. Steeg

A method is presented for synthesizing a type of control system designated as a final time controller. This device is a feedback control system that is designed to achieve a desired response at one time only, the response at earlier times being arbitrary within physical limits. In addition to a dynamic element which has a response that is to be controlled, the control loop consists of a feedback component and a controller, both of which have characteristics derived by a synthesis procedure.

The synthesis procedure presented results in a time varying, nonlinear system sufficiently simple so that optimization with respect to Gaussian random disturbances and to initial transients may be made easily. A time nonlinear optimum system is achieved, where the nonlinearity is saturation, the principal limitation for most control systems.

Application of the procedure reduces the analysis of the closed-loop system performance to the investigation of a single first-order differential equation involving the impulse response of the controlled element.

A POSITIONING SERVOMECHANISM WITH A FINITE TIME DELAY AND A SIGNAL LIMITER .... D. H. Evans

An idealization of a servomechanism which is used in a digital positioning circuit is analyzed. The loop consists of an ideal integrator, a finite time delay and a nonlinear signal limiting element. The nonlinear element in the physical system contains logical circuitry which converts the digital information into a continuously variable error.

An exact solution using contour integration is obtained for the idealized system to provide a standard with which solutions obtained by approximation techniques may be compared. In addition, a simple approximation solution and an error term are obtained.

Finally, curves are given which express the time necessary to zero in on the final position from a given initial position as a function of the loop parameters. These curves indicate that a minimum, noncritical settling time may be obtained by proper selection of loop components.

THE STABILIZATION OF NONLINEAR SERVOMECHANISMS ENCOUNTERED IN ANTENNA INSTRUMENTATION .... J. Bacon

This paper concerns the problem of compensating the loop-gain of instrument servos to provide uniform transient response over a wide dynamic operating range. Particular consideration is given to applications in the realm of antenna measurements.

A distinction is made between situations where the nonlinearity results from characteristics of the follow-up device and where it results from the loop-gain being functionally related to an external variable. In the former case the loop-

gain is made self-linearizing; in the latter, the gain is effectively equalized with an auxiliary logarithmic servo. This necessarily requires a suitable slave potentiometer in nonlinear loop, actuated by the logarithmic servo.

Certain advantages accure from using a ladder attenuator as the slave component. These are described. It is shown how this combination logarithmic servo and ladder slave attenuator can make a loop-gain invariable which otherwise would be functionally related to a variable E by the equation $G = E^{\pm N}$. The idea is extended to the generation of polynominal terms of the same form as G.

ON THE DESIGN OF A-C NETWORKS FOR SERVO COMPENSATION .... H. Levenstein

An analytical method for analysis and synthesis of networks for AC servo compensation is presented. The response of a linear network to a modulated suppressed-carrier excitation is formulated in terms of in-phase and quadrature carrier components. The relationship between the modulation on these components and the exciting modulation is shown to depend upon operators simply related to the original network function.

The expansion of the data-frequency operators into partial fractions is shown to lead to a simple synthesis procedure for deriving the original network operator from the in-phase or quadrature operators. As an example, the process is applied to the derivation of a representative network for lead compensation of an AC servo responsive to the in-phase component of error signal. The use of RC networks in AC servo compensation is shown to be limited to derivative types of equalization.

FUNDAMENTAL EQUATIONS FOR THE APPLICATION OF STATISTICAL TECHNIQUES TO FEEDBACK CONTROL SYSTEMS .... G. A. Biernson

Basic equations necessary for applying statistical techniques to the design of control systems are presented clearly and concisely to make them more understandable and useful to the engineer.

A discussion of the difference between noise and signal inputs and the need for statistical techniques is followed by Gaussian expressions for noise amplitude density, the probability of noise saturation of system components and practical considerations in determining proper components saturation levels. After describing autocorrelation functions of input and output, the corresponding cross-correlation functions and their relation to rms values are derived, and the relation between statistical inputs to a system and the resulting rms output is developed by transient and by transform methods.

Appendices show graphical interpretations of convolution and correlation integrals, the relations between spectral density and time function transforms and a method for using an analog computer to generate mean-square values of system outputs for statistical inputs.

A SURVEY OF TECHNIQUES FOR THE ANALYSIS OF SAMPLED-DATA CONTROL SYSTEMS .... G. J. Murphy and R. D. Ormsby

The present use in control systems of pulsed-data links, track-while-scan radar and digital computers has stimulated interest in the analysis of sampled-data

control systems. The effort now being expended to develop techniques for analyzing such systems has resulted in several methods of analysis.

This paper presents a discussion of three principal methods of analysis currently used: the impulse response, frequency response and Z transformation of the system transfer functions. The methods are explained and applied, and the advantages and limitation of each are discussed.

FINAL VALUE CONTROLLER SYNTHESIS

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Summary -- A method is presented for synthesizing a type of control system designated as a final time controller. This device is a feedback control system that is designed to achieve a desired response at one time only, the response at earlier times being arbitrary within physical limits. In addition to the dynamic element, which has a response that is to be controlled, the control loop consists of a feedback element and a controller, both of which have characteristics derived by the synthesis procedure. The final time controller developed is a time varying, nonlinear system and the procedure is one of the few that are applicable to the synthesis of nonlinear controllers. Application of the procedure reduces analysis of the performance of the closed-loop system to the integration of a single, first order differential equation involving the impulse response of the controlled element.

DESCRIPTION OF FINAL VALUE CONTROLLERS

This paper presents a method for synthesizing a class of feedback control systems, which differ from ordinary servomechanisms in that the controlled system response or output is to assume a desired value at one particular time only, the response at earlier times being arbitrary. Systems of this type are designated here as final value controllers. An example of final value control is return-to-base operation of an aircraft. Efficient procedure requires that the aircraft approach its landing field in such a way as to pass through a ground-control-approach (GCA) gate at a particular time but the trajectory flown at previous times can vary within wide limits.

Many of the existing methods for the design of systems whose effectiveness depends on satisfying final conditions utilize ordinary servomechanism techniques[1-3] which attempt to zero an error signal at all times. However, the unnecessary restriction of minimizing an error weighted equally for all times may unnecessarily complicate the design procedure and reduce seriously the performance achievable.

The synthesis procedure presented here results in a sufficiently simple time varying, nonlinear system so that optimization with respect to Gaussian random disturbances and to initial transients may be made easily. A true nonlinear optimum system is achieved, where saturation in the prime mover is the nonlinear limitation. This factor is the principal nonlinear limitation for most control systems.

For the purpose of explanation, one particular final value control situation is considered and is idealized into the system shown in Fig. 1. For this system

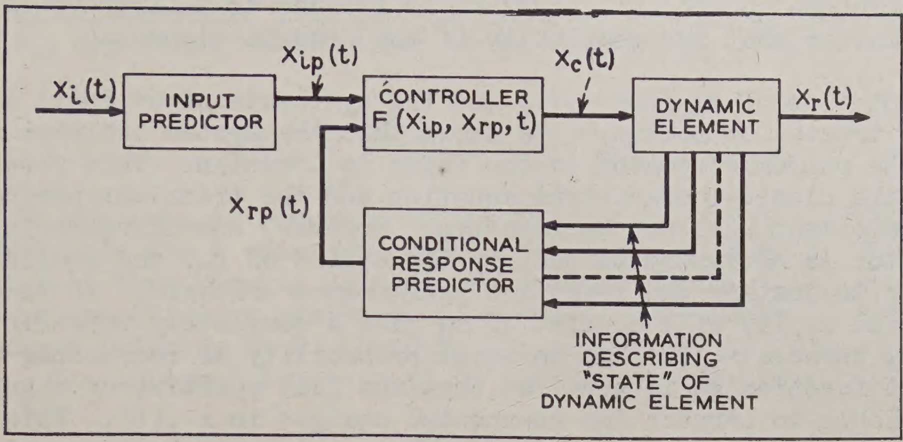

Fig. 1 - Final value controller.

the final value problem will be defined, the operation of the system described physically and the optimum characteristics pointed out.

STATEMENT OF THE PROBLEM

The problem consists of so synthesizing the several elements of the control system that the final response of the dynamic element $x_r(T)$ will equal some desired value x_d, the time T being the final time. The desired value x_d is related statistically to the input $x_i(t)$ with the result that x_d can be predicted from $x_i(t)$; e.g., $x_i(t)$ might be x_d plus noise. The given data for the problem consist of the input $x_i(t)$ and the dynamic element that is to be controlled. The synthesis process gives specifications for the operation of the input predictor, the controller and the response predictor through minimization of the final error E where

$$E = x_d - x_r(T). \tag{1}$$

The dynamic element is assumed to saturate at its input but otherwise to be linear. Thus, for the magnitude of $x_c(t)$ less than the saturation value x_{cm} the response $x_r(t)$ is related to $x_c(t)$ by a linear differential equation and for $|x_c(t)| > x_{cm}$ the effective input to the linear portion of the dynamic element is $\pm x_{cm}$. This statement constitutes the definition of the types of dynamic elements that are considered.

The physical operation of the system can be described as follows: at any time t the input predictor examines all past values of the input x_i and from these values makes the best prediction of the final value x_d. This prediction is designated $x_{ip}(t)$. At this same instant of time the conditional response predictor tells what will be the final value of the response if the control signal from t till the final time is some known function g_n. This conditionally predicted response is designated $x_{rp}(t)$. If $x_{rp}(t)$ equals $x_{ip}(t)$, then the controller acts to make g_n the control signal. If $x_{rp}(t)$ equals $x_{ip}(t)$, then the controller applies the maximum control signal consistent with saturation in the dynamic element in order to reduce the predicted terminal error.

Two properties achieved by the system are: (1) the prediction of x_d from $x_i(t)$ is completely separated from the closed-loop control of the dynamic element

and (2) the closed-loop-control equation is reduced to a first order differential equation no matter what the complexity of the dynamic element.

The prediction of x_d from $x_i(t)$ and the synthesis of an ideal input predictor has been treated by Booton.[4] He shows that the system achieves a minimum rms error when the random component in the input is Gaussian. This paper is concerned with the closed-loop-control equation and the transient performance. Booton's development will not be repeated. However, one characteristic of the input predictor is necessary to justify the choice of g_n, the equilibrium control function, and to justify the transient performance criteria. If the input predictor is ideal $x_{ip}(t)$ will consist of x_d plus a completely unpredictable noise. Consequently, because $x_{ip}(t)$ has an equal probability of increasing or decreasing, the best function g_n is zero, so that the full positive or negative control will be available to correct for unexpected changes in $x_{ip}(t)$. This hueristic argument can be made exact when $x_i(t)$ equals x_d plus a Gaussian random noise.

Because the variation in $x_{ip}(t)$ consists of an unpredictable noise, there can be no predictable transients in $x_{ip}(t)$ such as terms which vary as t or t^2. Consequently, the only significant transient response is that to an initial step change in $x_{ip}(t)$.

The principal assumptions and simplifications which have been used in the system of Fig. 1 are the following:

1. All statistical uncertainty is associated with $x_i(t)$. The response and state of the dynamic element are assumed to be exactly known so that if the control input were g_n, the conditionally predicted response would be exactly achieved. This assumption amounts to neglecting uncertainty about the nature of the dynamic element and noise in the measurement of its state. In some cases this uncertainty can be transferred to $x_i(t)$.

2. Only one final error, E, is minimized. Systems which simultaneously minimize a number of final conditions can also be developed.

3. The dynamic element is assumed to saturate at its input as previously discussed.

The action of the closed-loop system which has been described above is analyzed in the next section.

CLOSED-LOOP RESPONSE EQUATION

To develop the differential equation which describes the performance of the closed-loop system the operation of the controller and the conditional response predictor must be characterized.

The controller may be specified by a nondifferential function F,

$$x_c(t) = F\left[x_{ip}(t),\ x_{rp}(t),\ t\right] \tag{2}$$

which relates $x_c(t)$ to $x_{ip}(t)$, $x_{rp}(t)$, and t. The function will be in such form that $x_c(t)$ never exceeds the saturation limit x_{cm}. Usually F will have the

properties of an off-on relay type of controller as will be shown. In order to determine $x_{rp}(t)$, the final response may be written in the form

$$x_r(T) = \int_{-\infty}^{T} h(T-\tau)\ x_c(\tau)\ d\tau \tag{3}$$

in which $h(\tau)$ is the impulse response of the unsaturated dynamic element. Equation (3) applies only for a constant-coefficient dynamic element. However, the generalization to a time varying element can easily be made. For simplicity, only the constant-coefficient system is discussed. The integral in Eq. (3) extends from minus infinity to T. Because the controller is started at some finite time T_1, the integral from $-\infty$ to T_1 implicitly specifies the initial conditions at T_1.

At time t the conditional response predictor assumes that $x_c(t)$ becomes and remains a known function g_n which in the present example is taken to be zero. Thus, because $x_c(\tau)$ is taken as zero for $t<\tau<T$, the conditional response $x_r(T)\big|_{c,t}$ which is also the conditional predicted response may be written:

$$x_r(T)\big|_{c,t} = x_{rp}(t) = \int_{-\infty}^{t} h(T-\tau)\ x_c(\tau)d\tau. \tag{4}$$

The differential equation for the closed-loop system is obtained by differentiating Eq. (4) to yield

$$\frac{dx_{rp}}{dt} = h(T-t)\ x_c(t) \tag{5}$$

and combining this relation with Eq. (2) to yield

$$\frac{dx_{rp}}{dt} - h(T-t)\ F[x_{ip}(t),\ x_{rp}(t),\ t] = 0 \tag{6}$$

which is the desired equation describing the performance of the closed-loop system. Equation (6) can be integrated to obtain x_{rp}. Since at the final time, $x_{rp}(T) = x_r(T)$, the error E may be written

$$E = x_d - x_{rp}(T) \tag{7}$$

and a solution for E may always be obtained from Eq. (6). Usually this error is sufficient to evaluate a controller. However, the complete behavior of the system can be computed from $x_{rp}(t)$. Thus, Eq. (6) completely describes the final value performance of the system. In addition, it is a simple, first order equation.

Most of the desirable properties of final value controllers arise from the first order character of Eq. (6). Some of these properties can be illustrated by specifying a particular control function as shown in Fig. 2 and given by the equation

$$x_c(t) = \begin{cases} x_{cm} & \text{+ saturation region} \\ K\left[x_{ip}(t) - x_{rp}(t)\right] & \text{linear region} \\ -x_{cm} & \text{- saturation region.} \end{cases} \tag{8}$$

When the gain K becomes infinite an initial transient is removed in a minimum time and a minimum rms error due to Gaussian noise is achieved. The transient properties will now be developed.

In the absence of noise and with an ideal input predictor $x_{ip}(t)$ is constant and equal to x_d. Hence, the transient behavior is determined solely by the time necessary for $x_{rp}(t)$ to change from its initial value to some specified constant. This time is indicated most directly by Eq. (5). If the predicted terminal error is nonzero $x_c(t)$ will be $\pm x_{cm}$ and

$$\frac{dx_{rp}}{dt} = h(T-t)\ (\pm x_{cm}). \tag{9}$$

If

$$h(t) \geq 0* \tag{10}$$

for all values of t then Eq. (9) can be interpreted as showing that $x_{rp}(t)$ increases or decreases monotonically at the maximum rate as limited by the saturation level x_{cm} and the impulse response of the dynamic element h(t). Thus the initial error is reduced to zero for a minimum value of T as compared with any other possible controller.

If K is finite and x_{cm} sufficiently large that the system operates linearly then in the transient case Eq. (6) becomes

$$\frac{dx_{rp}}{dt} + Kh(T-t)\ x_{rp}(t) = Kh(T-t)\ x_d, \tag{11}$$

which can be integrated explicitly and the error evaluated at the final time to give

$$E = x_d - x_{rp}(T) = [x_d - x_{rp}(o)]\ \exp\left[-K\int_o^T h(t)dt\right] \tag{12}$$

in which the quantity $[x_d - x_{rp}(o)]$ is the final error if no control has been applied.

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*If h(t) does not satisfy condition (10) then the same optimum behavior can be obtained by a controller whose gain changes sign each time h(T-t) changes sign.

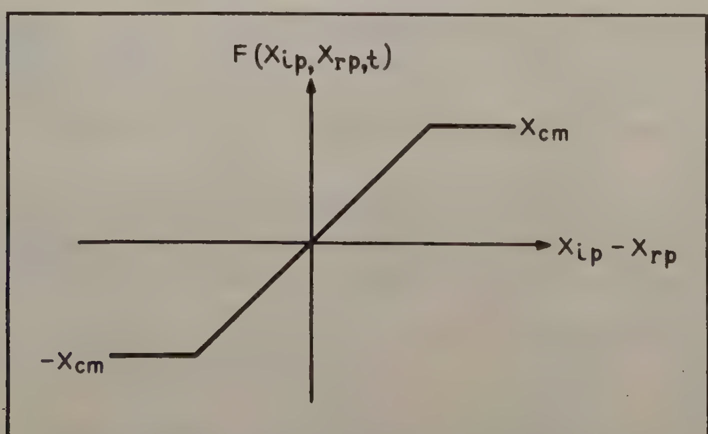

Fig. 2 - Nonlinear controller.

Equation (12) shows that if condition (10) is again satisfied the terminal error will decrease monotonically as the gain K is increased and as the terminal time T is increased. Thus the controller can be said to be stable for any value of gain.

This section has pointed out that the essential performance of the control system can be calculated by integrating one first order differential equation, Eq. (6). In addition, the optimum transient performance and stability of the system was indicated by an examination of the differential equation with one particular control function. Thus, the essential properties of the final time controller have been put in evidence. There remains to discuss the form of the conditional response predictor and present an illustrative example.

SYNTHESIS OF THE CONDITIONAL RESPONSE PREDICTOR

The function of the conditional response predictor has been specified in the preceding sections. At time t the predictor must compute the response of the dynamic element at the terminal time T under the condition that the control signal $x_c(t)$ is zero. Because the control signal is assumed to be zero during the interval from t to T, the response prediction is simply an evaluation of a homogeneous solution of the dynamic element differential equation.* This homogeneous solution must satisfy at time t a set of initial conditions which are by definition the state of the dynamic element. A set of state variables are defined as those that are sufficient to specify all the arbitrary constants in the complete homogeneous solution. For example, if the dynamic element differential equation is of the form

$$a_n\, x_r^{(n)}(t) + a_{n-1}\, x_r^{(n-1)}(t) + \dots + a_o\, x_r(t) = x_c(t) \tag{13}$$

then the response $x_r(t)$ and its n-1 derivatives $x_r^{(1)}(t), \dots x_r^{(n-1)}(t)$ make up a complete set of state variables. For this example, the conditional response prediction is most simply written in terms of the homogeneous solutions $f_i(t)$, $i = 0, 1, \dots, n-1$ which have the initial values

$$f_i^{(j)}(0) = \begin{cases} 1, i = j \\ 0, i \neq j \end{cases} . \tag{14}$$

With these functions the homogeneous solution having the state variables as its initial condition at time t is simply

$$F(u) = \sum_{i=o}^{n-1} x_r^{(i)}(t)\, f_i(u - t) . \tag{15}$$

In Eq. (15) u has been introduced as a new independent time variable to avoid confusion with t. That F(u) satisfies the required initial conditions can be established by taking its j^{th} derivative and setting u = t which as a result of Eq. (14) yields

$$F^{(j)}(t) = \sum_{i=o}^{n-1} x_r^{(i)}(t)\, f_i^{(j)}(o) = x_r^{(j)}(t) . \tag{16}$$

*In the case of a nonzero g_n function the prediction would be the sum of the homogeneous solution and a known particular solution.

The conditional response at the terminal time is simply F(T) and thus

$$F(T) = x_{rp}(t) = \sum_{i=0}^{n-1} x_r^{(i)}(t)\, f_i(T-t) \; . \tag{17}$$

Equation (17) can be interpreted physically to synthesize the predictor. The response $x_r(t)$ and its derivatives form the inputs to the predictor and the functions $f_i(T-t)$ are generated within the predictor. The entire predictor is realized as shown in Fig. 3. If the dynamic element differential equation is more complicated than Eq. (13) then an expression as simple as Eq. (16) may be obtainable for the conditional response predictor. However, the general procedure of finding a homogeneous solution and a complete set of initial conditions can always be used to construct the predictor.

The discussion of the example gives the false impression that $x_r(t)$ and its derivatives are usually measured as state variables in a physical element. This procedure is seldom desirable since the higher derivative measurements will be

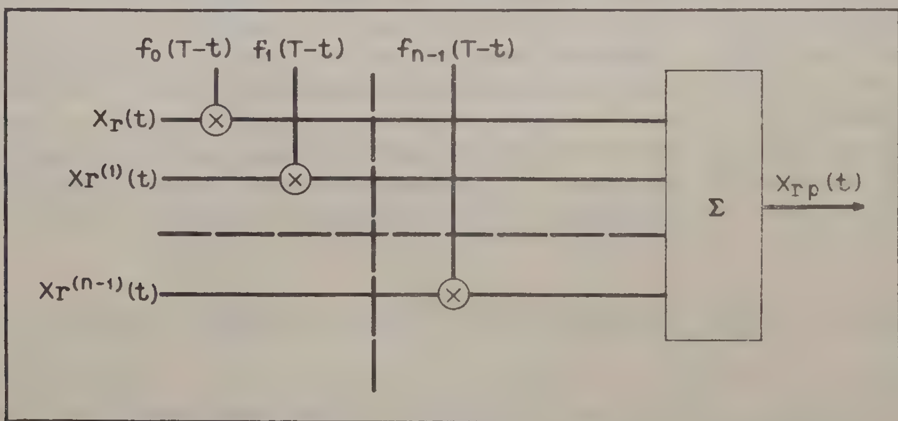

Fig. 3 - Conditional response predictor.

affected by noise. Instead, for almost all cases a set of easily measurable state variables can be determined as the energy stored in various forms in the dynamic element.

ANALYSIS OF AN EXAMPLE

A realistic yet elementary example with which to illustrate the final value control theory is difficult to obtain. The theory was developed too recently for many systems to have been examined and most of the applications deal with complicated controllers which satisfy more than one terminal condition. As a consequence a simple hypothetical system will be presented, the conditional response predictor equations developed and the transient response computed.

The control system is shown in Fig. 4 and consists of the usual elements: input predictor, controller, dynamic element and conditonal response predictor. The dynamic element saturates at a control-signal magnitude x_{cm}. The unsaturated element consists of a simple time constant followed by an integration and is described by the equation

$$\tau x_r''(t) + x_r'(t) = x_c(t). \tag{18}$$

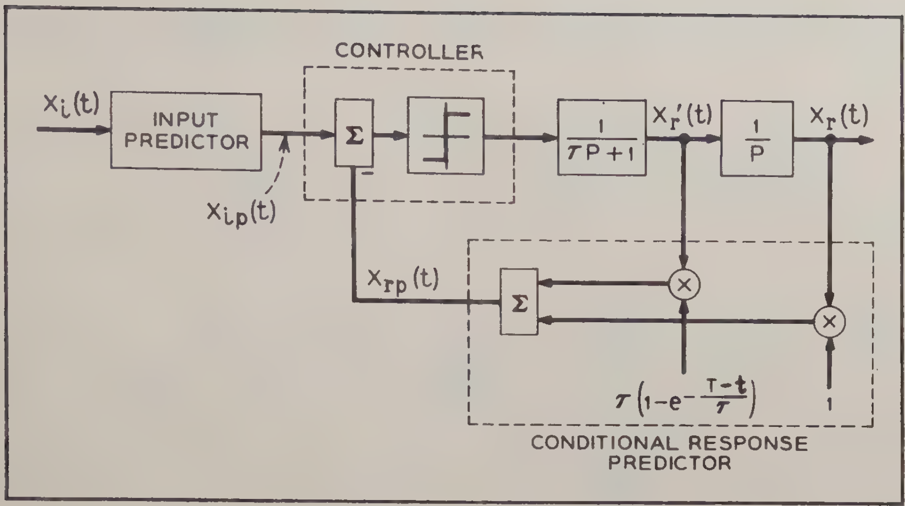

Fig. 4 - Simple final value controller.

The controller is assumed to be the ideal controller shown in Fig. 2 with an infinite K. The conditional response predictor equations are developed below. We will assume the input predictor to be ideal so that to evaluate the transient performance in the absence of noise $x_{ip}(t) = x_d$. The impulse response of the dynamic element may easily be shown to be

$$h(t) = 1 - e^{-\frac{t}{\tau}} \tag{19}$$

and the homogeneous response of the dynamic element to be

$$x_r(t) = A + Be^{-\frac{t}{\tau}} \tag{20}$$

where A and B are arbitrary constants of integration.

To determine the conditional response predictor equations the state variable must be specified first. These variables are simply $x_r(t)$ and its derivative $x_r'(t)$. The homogeneous response of the dynamic element with initial condition $x_r'(t)$, $x_r(t)$ can either be computed from a direct analysis of the dynamic element or from Eq. (20). With either procedure the response $f_o(t)$ corresponding to $x_r(0) = 1$, $x_r'(0) = 0$ is

$$f_o(t) = 1 \tag{21}$$

and the response $f_1(t)$ corresponding to $x_r(0) = 0$, $x_r'(0) = 1$ is

$$f_1(t) = \tau(1 - e^{-\frac{t}{\tau}}) \ . \tag{22}$$

Thus, the conditional predicted response at time T corresponding to $x_r(t)$, $x_r'(t)$ at time t is

$$x_r(T)\Big|_{c,t} = x_{rp}(t) = x_r(t) + x_r'(t)\ \left[\tau - \tau e^{-\frac{T-t}{\tau}}\right] . \tag{23}$$

The conditional response predictor shown in Fig. 4 is an instrumentation of Eq. (23). Thus, the synthesis of the control system is complete.

The terminal error can be evaluated as $x_d - x_{rp}(T)$ by integrating Eq. (24) from 0 to T to yield

$$E = x_d - x_{rp}(T) = x_d - (T + \tau e^{-\frac{T}{\tau}} - \tau)\, x_{cm} \,, T \leq T_c. \tag{25}$$

Equation (25) specifies the terminal error E for values of T less than some critical value T_c where the error becomes zero. For $T > T_c$ the final error is always zero.

A plot of normalized final error E/x_d is shown in Fig. 5 as a function of normalized time T/τ and for several values of saturation x_{cm}. In Fig. 6 the final value control system performance is compared with that of a well-adjusted linear servo using the same dynamic element. The block diagram of the servo is also sketched in Fig. 6. The ratio of x_d to x_{cm} has been so selected that the maximum value of x_c in the linear servo just equals x_{cm}. As demonstrated in Fig. 6, the final value controller not only achieves a zero error for a smaller

The transient response can be evaluated by assuming the conditions $x_r(0) = x_r'(0) = 0$, and $x_{ip}(t) = x_d$. The final error can then be computed as a function of the final time T. The error will decrease to zero as T is increased. In the region of nonzero terminal error $x_d - x_{rp}(t)$ will also be nonzero, $0 < t < T$, and the controller will apply the maximum input x_{cm} to the dynamic element during the entire trajectory. Thus, from Eq. (9)

$$\frac{dx_{rp}}{dt} = (1 - e^{-\frac{T-t}{\tau}})\, x_{cm} \tag{24}$$

value of T than required by the linear servo but also the error remains zero for $T > T_c$, while the linear system has a substantial overshoot.

CONCLUSION

A procedure has been developed for the synthesis of certain time varying, nonlinear systems. The theory tends to show the advantages of these systems over

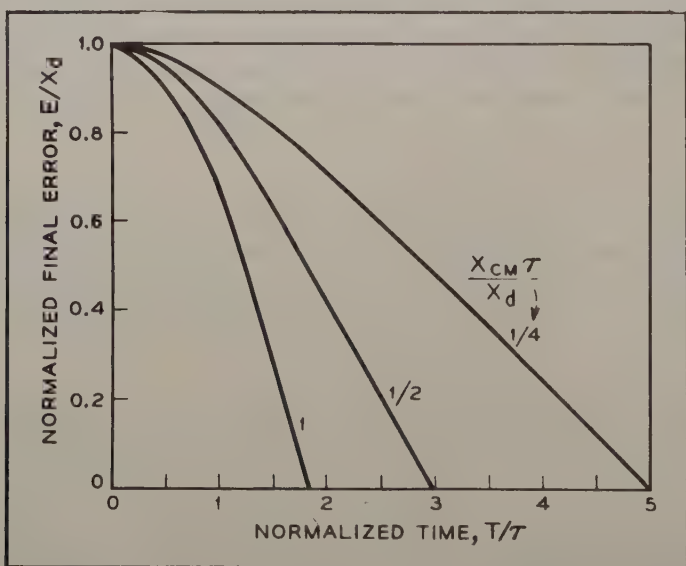

Fig. 5 - Final value controller performance.

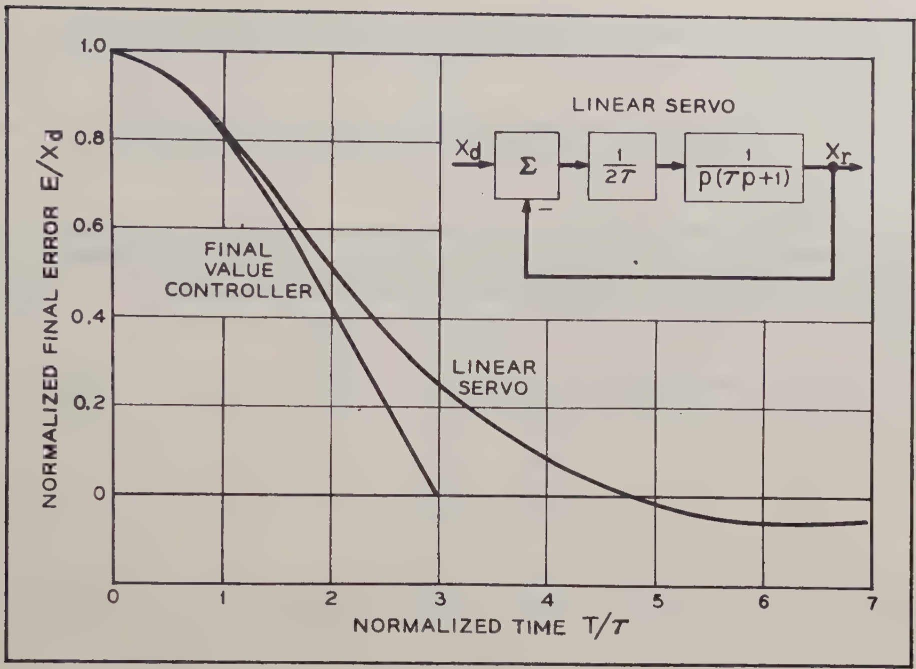

Fig. 6 - Comparison of linear servo and final value controller.

systems designed by previously published methods. In some cases this synthesis procedure results in the best possible performance obtainable with a particular desired response and with a given dynamic element. The procedure is an example of a synthesis involving nonlinear differential equations and is simpler than most analytical methods for such equations. This simplicity is achieved because the synthesis procedure is chosen to result in an analytically tractable system (specifically to result in a first order system). Consequently, the requirement of tractability has been added to the usual synthesis requirement of good system performance. Tractability may be the factor that has limited the use of nonlinear synthesis schemes.

Several direct extensions of the procedure which were not discussed are apparent. The final value controllers were designed to satisfy only one terminal condition, namely, that the response of the dynamic element at the terminal time equal some desired value. Systems that satisfy additional requirements involving the response and its derivatives at one or more terminal times can be developed. The assumption was made that the equilibrium value of the control input to the dynamic element is zero. Other controllers can be derived in which the equilibrium value can be any function of time.

The synthesis procedure is applicable to dynamic elements which are linear but saturable. While these elements undoubtedly form a very important class, the synthesis procedure probably can be extended to include other types of dynamic nonlinear elements.

BIBLIOGRAPHY

1. Hall, A. C.: "The Analysis and Synthesis of Linear Servomechanisms." Cambridge, Massachusetts, The Technology Press, MIT, 1943.

2. Evens, W. R.: "Graphical analysis of control systems." Trans. AIEE, 67:547-551 (1948).

3. Truxal, J. G.: "Automatic Feedback Control System Synthesis." New York, McGraw-Hill Book Company, Inc., 1955.

4. Booton, R. G.: "Optimum design of final-value control systems." Microwave Research Institute, Polytechnic Institute of Brooklyn, Proceedings of Symposium on Nonlinear Circuit Analysis, Vol. VI (1956).

A POSITIONING SERVOMECHANISM WITH A FINITE TIME DELAY AND A SIGNAL LIMITER

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Summary -- An idealization of a servomechanism which is used in a digital positioning circuit is analyzed. The loop consists of an ideal integrator, a finite time delay and a nonlinear signal limiting element. An exact solution is derived. Also, a simple approximate solution and an error term are obtained. Finally, curves are given which express the time necessary to zero in on the final position from a given initial position as a function of the loop parameters.

INTRODUCTION

There are two salient features about the model to be analyzed -- the nonlinear signal limiting element and the finite time delay. Although the physical system which led to this analysis will not be considered we will indicate the sources of these features.

The nonlinear element derives from the fact that the physical system is a digital positioning servomechanism. The nonlinear element in the physical system contains logical circuitry which converts the digital (and transitional digital) information into a continuously variable error. The model actually has the nonlinear element at a different place in the loop than is indicated by the above; this was for ease of analysis and is immaterial to the applicability of the analysis.

The finite time delay arose from the high speed at which the physical positioning servomechanism operates. The positioning speed relative to the finite speed of propagation of signals through the system made the delay due to signal propagation noninfinitesimal. In the model the delays are lumped.

THE LOOP EQUATIONS FOR THE MODEL

In the model shown in Fig. 1. the feedback loop has unity gain. The box labeled Δ contributes a delay of Δ sec; that is, if the input is $v(t+\Delta)$, the output is $v(t)$. f is a nonlinear device such that if $v(t)$ is the input $f[v(t)]$ is the output. A is an amplifier. $\theta_i(t)$ and $\theta_o(t)$ are the input and output, respectively.

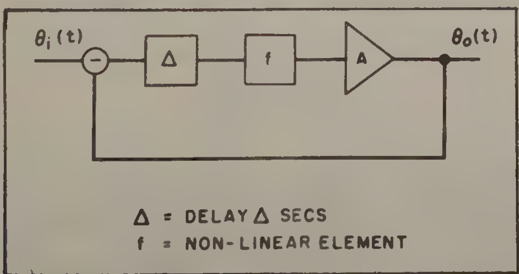

Fig. 1

For a step function input of amplitude M at time $t = 0$, and $\Theta_o(0) = 0$, i.e., an initial error M,

$$u(t + \Delta) = M - \int_0^t f[u(x)]\, K(t - x)\, dx, \quad t > 0 \tag{1}$$

$$u(t) = M \qquad 0 < t < \Delta$$

where

$$u(t) = \Theta_i(t) - \Theta_o(t) = \text{error}$$

$$K(t) = \delta\text{-function response of amplifier.}$$

The particular nonlinear f which we use is shown in Fig. 2. Analytically,

$$f[u(t)] = \begin{cases} \gamma, & u(t) > \lambda \\ \frac{\gamma}{\lambda} u(t), & |u(t)| \leq \lambda \\ -\gamma, & u(t) < -\lambda \end{cases}$$

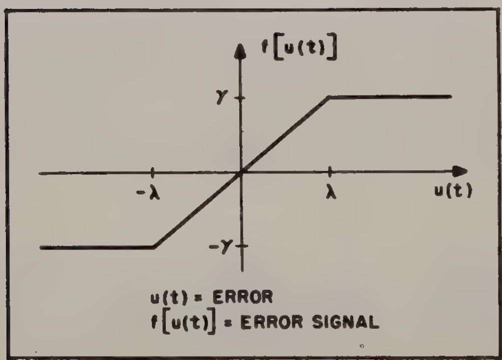

Fig. 2

The particular amplifier we wish to consider is one with a δ-function response

$$K(t) = A_o \beta e^{-\beta t}.$$

K(t) can be approximated very closely for this problem by an ideal integrator. Hence, take

$$K(t) = A_o \beta . \tag{2}$$

With the above assumptions a qualitative picture of the response is shown in Fig. 3. The diagram illustrates why we can use an integrator to approximate the actual amplifier. The requirements are that the range 2λ be sufficiently small so that the characteristic is nearly linear within this range, and that the equilibrium voltage must be large compared to M so that the characteristic is almost linear up to the linear range of f. (This last requirement is not too

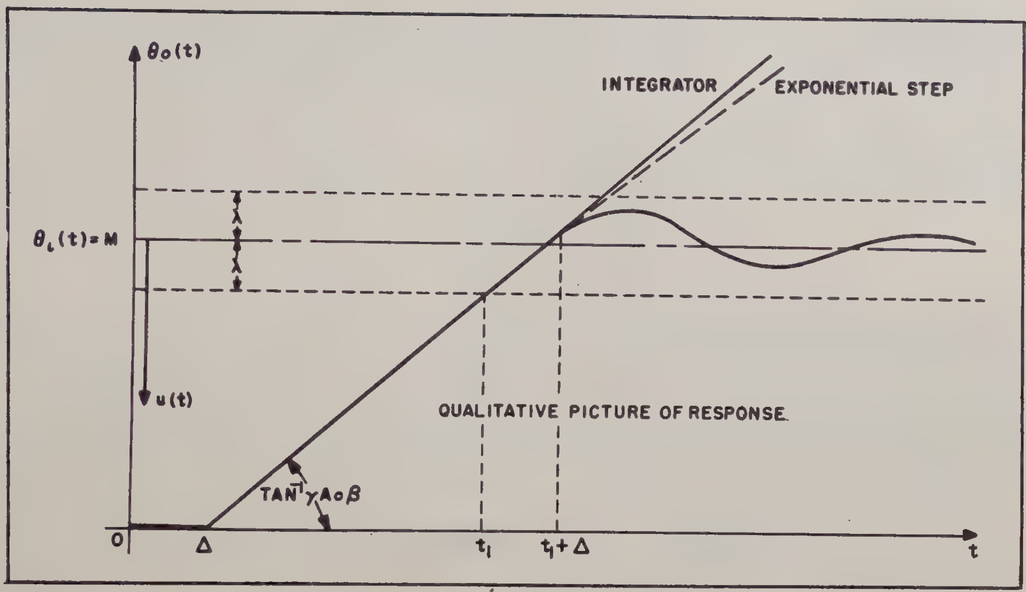

Fig. 3

important; it will be obvious to the reader as he reads on how the nonlinearity could be taken care of.)

It is desirable to put the problem in nondimensional form. Let

$$\left\{\begin{aligned} & t = \Delta\xi \\ & \lambda r(\xi) = u(t) \\ & \frac{\gamma A_o \beta\Delta}{\lambda} = \mu \end{aligned}\right. \tag{3}$$

Then

$$\frac{1}{\gamma} f[u(t)] = \frac{1}{\gamma} f[\lambda r(\xi)] = h[r(\xi)] = \begin{cases} 1, & r(\xi) > 1 \\ r(\xi), & |r(\xi)| \leq 1 \\ -1, & r(\xi) < -1 \end{cases}$$

Hence, inserting Eqs. (2) and (3) into (1), one obtains

$$\begin{aligned} r(\xi + 1) &= \frac{M}{\lambda} - \mu \int_0^{\xi} h[r(x)]\, dx, \quad \xi > 0 \\ r(\xi) &= \frac{M}{\lambda}, \qquad\qquad 0 < \xi < 1. \end{aligned} \tag{4}$$

THE SOLUTION OF THE LOOP EQUATION

From Fig. 3 it is obvious that the response of the system breaks up naturally into two parts. The first part is the slewing mode where the error, u(t),

decreases linearly from M to λ, after an initial time lag Δ, or in the nondimensional formulation the error, $r(\xi)$, decreases linearly from M/λ to unity after an initial time lag of unity. The second part starts after the error signal has entered the linear region of f or h.

The first part is easily calculated from Eq. (4). One finds that

$$\xi_1 = \frac{M/\lambda - 1}{\mu} + 1 \tag{5}$$

where ξ_1 is the time at which the signal first enters the linear region, ξ_1 corresponds to t_1 in Fig. 3.

Because of the time delay, the error signal, $r(\xi)$, continues to decrease linearly up to time $\xi = \xi_1 + 1$. At $\xi = \xi_1 + 1$ the error signal has penetrated the linear region of r by the amount μ. This is shown in Fig. 4.

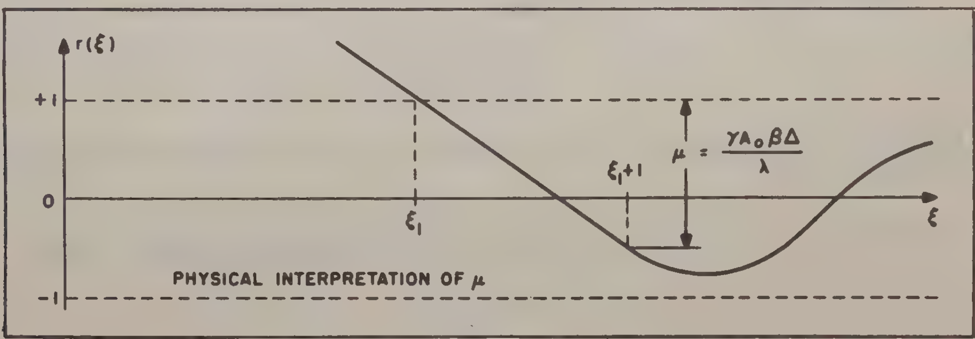

Fig. 4

We can now proceed to solve the second part. We initially restrict ourselves to the special case $|r(\xi)| \leq 1$ for $\xi \geq \xi_1$, i.e., once the error signal enters the linear region it does not leave it again. Redefining our time origin so that $\xi = \xi_1$ corresponds to $\tau = 0$ and $r(\xi) = w(\tau)$, Eq. (4) becomes

$$\begin{aligned} w(\tau + 1) &= (1 - \mu) - \mu \int_0^\tau w(x)\, dx \qquad \tau > 0 \\ w(\tau) &= 1 - \mu\tau \qquad 0 \leq \tau < 1 \end{aligned} \tag{6}$$

One takes the Laplace transform of Eq. (6) using the boundary condition on $w(\tau)$ for $0 \leq \tau < 1$. Taking the inversion integral one finds

$$w(\tau) = 1 - \frac{\mu}{2\pi i} \int_{-i\infty}^{+i\infty} \frac{e^{s(\tau + 1)}}{s(se^s + \mu)}\, ds, \qquad \tau > 0. \tag{7}$$

We now proceed to evaluate the above integral. One can show that the above integral is equivalent to a contour integral in the s-plane. The contour is a rectangle with corners at $(0, -iN\pi)$, $(0, iN\pi)$, $(-N, iN\pi)$, $(-N, -iN\pi)$. As N goes

to infinity through odd positive integers the contour integral approaches the above integral (Appendix I). Hence, one can use the residue theorem.

The zeros of the denominator are at $s = 0$ and the zeros of $g(s) = se^{s} + \mu$. The pole at $s = 0$ simply cancels out the first term, 1, in Eq. (7). In order to find the zeros of $g(s)$, let $s = \xi + i\eta$; after some manipulation one finds

$$-\xi = \eta \cot \eta, \qquad \sin \eta \neq 0 \tag{8}$$

$$\xi e^{\xi} = -\mu \cos \eta, \qquad \cos \eta \neq 0 \tag{9}$$

A few of these curves are plotted in Fig. 5 for various values of μ and the zeros are circled. Of course, the families continue indefinitely in both the

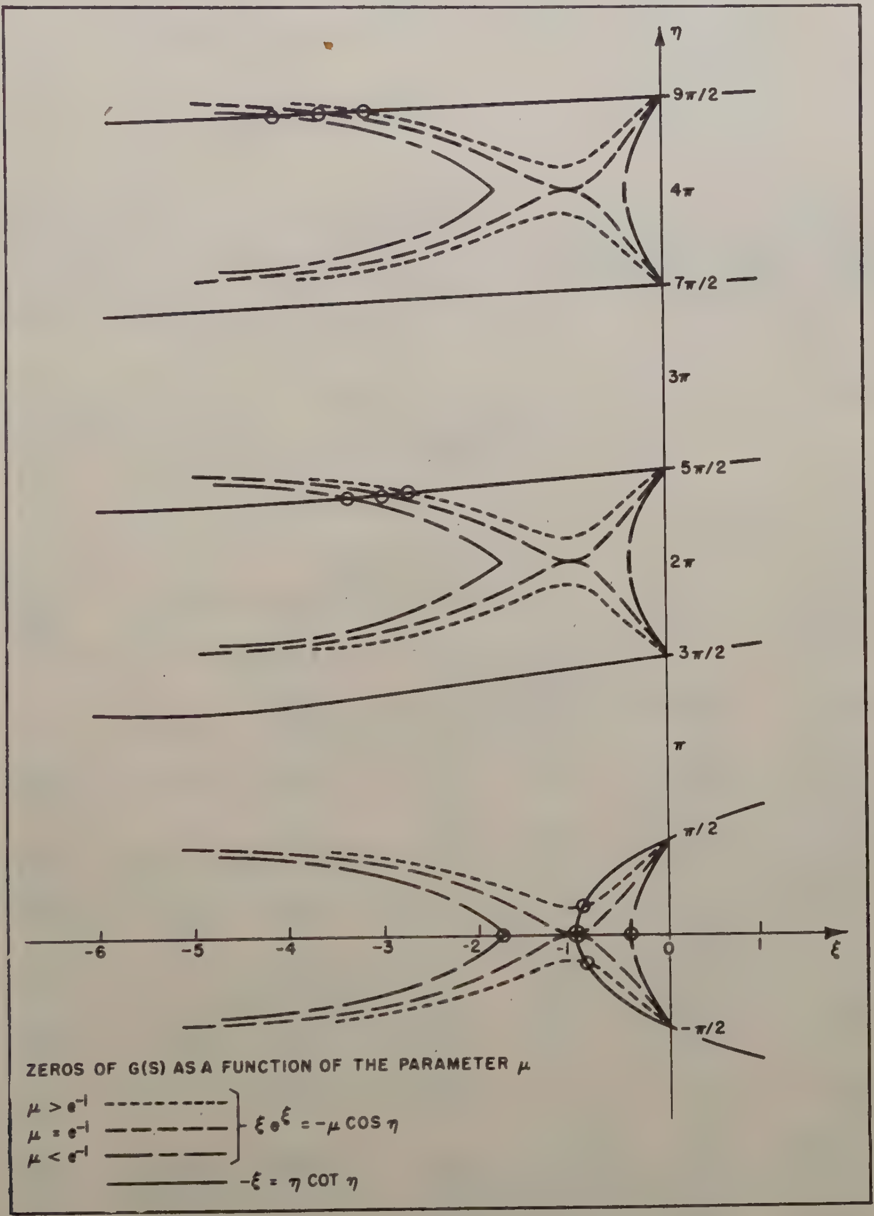

Fig. 5

positive imaginary and negative imaginary directions. The apparent zeros on the imaginary axis do not exist because of violations of the conditions on Eqs. (8) and (9). The zeros are all simple except for the special case $\mu = e^{-1}$, $\xi = -1$, which is a double zero. One finds that the system is stable for

$$\mu < \pi/2 = 1.571. ^{1} \tag{10}$$

For $|\eta| \geq 2\pi$ and for μ sufficiently large, say $\mu \geq 0.7$, one can get very good approximations to the roots by linearizing Eqs. (8) and (9).

$$\mathrm{Re}(s_n) = \xi_n = -\ln(2n + 1/2)\pi + \ln\mu \tag{11}$$

$$\mathrm{Im}(s_n) = \eta_n = (2n + 1/2)\pi + \frac{\xi_n}{(2n + 1/2)\pi} \tag{12}$$

$$n = 1, 2, 3 \text{ ---}$$

The principal zeros, i.e., those for which $|\eta_0| \leq \pi/2$, can be obtained by solving Eqs. (8) and (9) simultaneously by successive numerical approximations. A perturbation expansion in powers of $(\pi/2 - \mu)$ is used to obtain the zero order approximation. Values for ξ_0, η_0 as functions of μ are given in Table I and are plotted in Fig. 6.

TABLE I

			$\delta = 0.2$				$\delta = 0.1$	
μ	$-\xi_0$	η_0	T_3	T_6	T_{12}	T_{25}	T^*_{12}	T^*_{25}
0.7	0.565	1.094	10.9	19.5	36.6	73.8	37.8	75.0
0.8	0.473	1.194	10.6	18.0	33.0	65.6	34.5	67.0
0.9	0.391	1.272	10.6	17.2	30.6	59.5	32.3	61.2
1.0	0.318	1.337	11.0	17.0	29.0	55.0	31.0	57.0
1.1	0.252	1.392	11.9	17.3	28.2	51.9	31.0	54.6
1.2	0.191	1.439	13.6	18.6	28.6	50.3	32.2	53.9
1.3	0.134	1.481	17.0	21.6	30.8	50.8	36.0	56.0
1.4	0.0802	1.517	25.1	29.4	38.0	56.6	46.6	65.2
1.5	0.0321	1.550	56.3	60.3	68.3	85.6	90.0	107.4

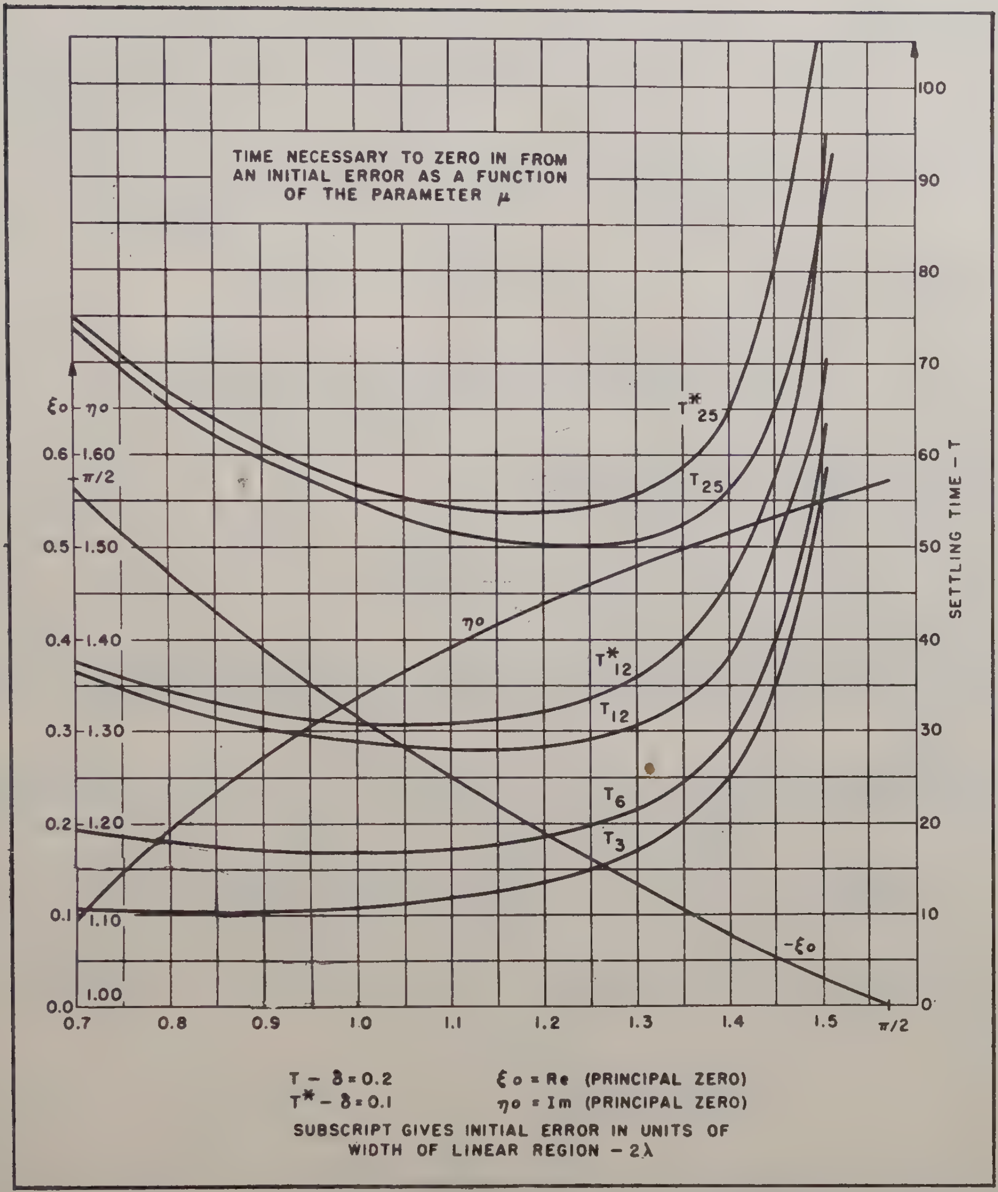

Fig. 6

Knowing the zeros of g(s) one can calculate the residues of the integral in Eq. (7). Letting

$$\varphi_i = \mathrm{Tan}^{-1} \frac{\eta_i}{\xi_i + 1}, \qquad \pi \geq \varphi_i \geq 0 \tag{13}$$

and

$$R_i = -\frac{2}{\mu} e^{\xi_i(\tau + 1)} \frac{\cos\left[\eta_i \tau + (\eta_i - \varphi_i)\right]}{\sqrt{(\xi_i + 1)^2 + \eta_i}}, \quad \eta_i \neq 0, \tag{14}$$

one finds three cases:

Case 1: $0 < \mu < e^{-1}$; two real zeros, call them a_1 and a_2; all others complex:

$$w(\tau) = \frac{e^{a_1(\tau + 1)}}{a_1 + 1} + \frac{e^{a_2(\tau + 1)}}{a_2 + 1} + 2 \sum_{n=1}^{\infty} R_n \tag{15}$$

Case 2: $\mu = e^{-1}$; one double real root; all others complex:

$$w(\tau) = 2e^{-(\tau + 1)} (\tau + 2) + 2 \sum_{n=1}^{\infty} R_n \tag{16}$$

Case 3: $e^{-1} < \mu < \pi/2$; all roots complex:

$$w(\tau) = 2 \sum_{n=0}^{\infty} R_n \tag{17}$$

The three cases correspond respectively to overdamping, critical damping and underdamping. We are interested primarily in Case 3.

For the underdamped case, Case 3, we can rewrite Eq. (17) in the form

$$w(\tau) = 2e^{\xi_0(\tau + 1)} \frac{\cos [\eta_0 \tau + (\eta_0 - \varphi_0)]}{\sqrt{(\xi_0 + 1)^2 + \eta_0^2}} + \varepsilon(\tau). \tag{18}$$

The error term, $\varepsilon(\tau)$, is bounded by (Appendix II)

$$|\varepsilon(\tau)| \leq 4 \left(\frac{\mu}{2\pi}\right)^{\tau + 1}, \quad \tau \geq 1. \tag{19}$$

$\varepsilon(\tau)$ becomes small very rapidly; one sees that for the worst case, $\mu = \pi/2$, that even at $\tau = 2$, $|\varepsilon(\tau)| \leq 0.06$.

The above analysis has been a linear analysis; it is valid only if $|w(\tau)| \leq 1$ for all $\tau > 0$. In order to find the range of μ for which the above is fulfilled one can go back to the defining integro-difference equation -- Eq. (6). This can be integrated piecewise, i.e., first for $0 \leq \tau \leq 1$, then for $1 \leq \tau \leq 2$, etc. Performing the indicated operations for $0 \leq \tau \leq 2$, then differentiating the resulting expressions for $w(\tau)$, one finds that $w(\tau)$ has a minimum at

$$\tau_m = \frac{1 + \mu}{\mu} \text{ if } 1 \leq \tau_m \leq 2 \tag{20}$$

and that, in this case,

$$w(\tau_m) = 1/2 - \mu. \tag{21}$$

These equations show that $|w(\tau_m)| \leq 1$ for $\mu \leq 1.5$, and hence, that the linear analysis is valid for $\mu \leq 1.5$. [Equations (18) and (19) show that $w(\tau)$ does not go out of the linear range at any subsequent maximum or minimum.]

We have thus shown that Eq. (18) is valid for all $\tau > 0$ for $\mu \leq 1.5$. Since, for stability, one must have $\mu < \pi/2 = 1.57$, the only range of interest for which the above analysis is invalid is $1.50 < \mu < 1.57$. This range is too small to be of engineering interest and we do not choose to consider it.

CONCLUSION

Finally, we compute the minimum time necessary for the error to become essentially zero after starting out at an initial value M as a function of the parameter μ. Let "essentially zero" mean that $w(\tau)$, as given by Eq. (18), must be less than or equal to δ for all $\tau \geq \tau_e$. We ignore the cosine contribution in computing τ_e. Then the time, T, in units of Δ, for the error to decrease to essentially zero from an initial error of M/λ, again in dimensionless units, is simply ξ_1, as given by Eq. (5) (i.e., the dimensionless time necessary for the error signal to get to the linear region) plus τ_e as given above. T is plotted as a function of μ for $\delta = 0.1$ and 0.2 and for $M/2\lambda = 3, 6, 12, 25$ in Fig. 6. The values are tabulated in Table I. The minimum can be obtained graphically or by examining the values of T. It is apparent from these curves that the minimums are shallow and therefore the value of μ is not critical.

The most important application of the results is not the engineering one. It is well known that there are various approximation techniques which can be used to obtain engineering answers for the system analyzed here. They can also be used to engineer more complicated systems of roughly the same nature. But the importance of the solution obtained is that it can be used as a check point for current approximation techniques and others which will be developed. The utility is enhanced by the simplicity of the approximate solution and the fact that it was possible to obtain a simple quantitative bound for the error.

APPENDIX I

Consider the integral

$$I(\tau) = \int_{\Gamma_N} \frac{e^{s(\tau+1)}}{s(se^s + \mu)}\, ds$$

where Γ_N is the rectangular path in the s-plane with vertices at $(0,-iN\pi)$, $(0,iN\pi)$, $(-N,iN\pi)$ and $(-N,-iN\pi)$ and which is suitably indented to enclose the origin. Let

$$I(\tau) = \int_{-iN\pi}^{+iN\pi} + \int_{iN\pi}^{-N+iN\pi} + \int_{-N+iN\pi}^{-N-iN\pi} + \int_{-N-iN\pi}^{-iN\pi}$$

$$= J_1 + J_2 + J_3 + J_4$$

It is obvious that J_1 approaches the integral we want to evaluate as $N \to \infty$.

Now consider J_2.

$$|J_2| \leq \left| \int_{iN\pi}^{-N+iN\pi} \frac{|e^{s(\tau+1)}|}{|s||se^s + \mu|} |ds| \right|$$

Let $s = x + iN\pi$, then

$$|se^s + \mu| \geq e^x |x \sin N\pi + N\pi \cos N\pi|$$

after dropping the imaginary part. Letting N be an odd (positive) integer, this becomes

$$|se^s + \mu| \geq e^x N\pi$$

and thus

$$|J_2| \leq \int_0^{-N} \frac{e^{x(\tau+1)}}{(N\pi)\, e^x N\pi} dx = \frac{1}{(N\pi)^2 \tau} |e^{-N\tau} - 1|$$

Therefore, for $\tau > 0$

$$\lim_{N \to \infty} J_2 = 0.$$

J_4 behaves in exactly the same manner.

Finally, consider J_3,

$$|J_3| \leq \left| \int_{-N+iN\pi}^{-N-iN\pi} \frac{|e^{s(\tau+1)}|}{|s||se^s + \mu|} |ds| \right|$$

Let $s = -N + iy$, then

$$|se^s + \mu| \geq |\mu - e^{-N}|N \cos y + y \sin y||$$

after dropping the real part. Since $|y| \leq N\pi$

$$e^{-N}|N \cos y + y \sin y| \leq e^{-N} N(\pi + 1)$$

Given a positive number $\delta < \mu$, there exists N_o such that for $N > N_o$

$$e^{-N} N(\pi + 1) \leq \mu - \delta$$

Hence,

$$|se^{s} + \mu| \geq \delta > 0$$

and therefore,

$$|J_3| \leq \int_{-N\pi}^{+N\pi} \frac{d^{-N(\tau + 1)}}{N\delta} \, dy = \frac{2\pi}{\delta} e^{-N(\tau + 1)}, \; N \geq N_o$$

Therefore, for $\tau \geq 0$

$$\lim_{N\to\infty} J_3 = 0$$

APPENDIX II

$$\varepsilon(\tau) = 2 \sum_{n=1}^{\infty} e^{\xi_n(\tau + 1)} \frac{\cos\, [\eta_n\tau + (\eta_n - \varphi_n)]}{\sqrt{(\xi_n + 1)^2 + \eta_n^2}}$$

Since one can satisfy himself that $\eta_n \geq 1$, $n = 1,2,\ldots.$, by looking at Fig. 5, then certainly

$$|\varepsilon(\tau)| \leq 2 \sum_{n=1}^{\infty} e^{\xi_n(\tau + 1)}$$

From Eq. (11)

$$|\varepsilon(\tau)| \leq 2 \sum_{n=1}^{\infty} \left[\frac{\mu}{(2n + \frac{1}{2})\pi}\right]^{\tau + 1} \leq 2\left(\frac{\mu}{2\pi}\right)^{\tau + 1} \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^{\tau + 1}$$

This series can be shown[2] to be bounded by

$$\sum_{n=1}^{\infty} \frac{1}{n}^{\tau + 1} \leq 1 + \int_{1}^{\infty} \frac{dx}{x^{\tau + 1}} = 1 + \frac{1}{\tau}, \; \tau > 0$$

Hence,

$$|\varepsilon(\tau)| \leq 4\left(\frac{\mu}{2\mu}\right)^{\tau + 1}, \quad \tau \geq 1$$

REFERENCES

1. This agrees with a result in "Stability criteria for an electrical or mechanical system with distributed parameters," A. S. Gladwin, B. J. Applied Physics, vol. 6, no. 11 (Nov. 1955). In his notation $C_0 = \mu$, $C_1 = 2$, $C_2 = -\mu$, $C_3 = 2$.

2. For instance, Courant, "Differential and integral calculus," Interscience, vol. I, p. 381.

THE STABILIZATION OF NONLINEAR SERVOMECHANISMS ENCOUNTERED IN ANTENNA INSTRUMENTATION*

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Summary -- This paper investigates the problem of compensating the loop-gain of instrument servos to provide uniform transient response over a wide dynamic operating range. Particular consideration is given to applications in the realm of antenna measurements. A distinction is made between situations where the nonlinearity results from characteristics of the follow-up device and where it results from the loop-gain being functionally related to an external variable. In the former case the loop-gain is made self-linearizing; in the latter the gain is effectively equalized with an auxiliary logarithmic servo. This necessarily requires a suitable slave potentiometer in the nonlinear loop, actuated by the logarithmic servo. Certain advantages resulting from the use of ladder attenuator as the slave component are described. It is shown how this combination logarithmic servo and ladder slave attenuator can make a loop-gain invariable which otherwise would be functionally related to a variable E by the equation $G = E^{\pm N}$. The idea is extended to the generation of polynominal terms of the same form as G.

INTRODUCTION

Automatic instruments employed in making antenna measurements are principally those which record either logarithmic or square-root functions of amplitude or linear functions of phase. The major design difficulty encountered in these instruments is that of providing for uniform servo response throughout the full dynamic range. In the usual linear servo this consideration is no problem because the loop-gain is constant. In antenna instruments, however, the loop-gain is often a function of a generalized variable u.

Figure 1 shows a typical situation where the follow-up device is functionally related to the variable u. For certain types of amplitude plotters u is identified with the output-shaft position. In other cases, say for phase plotters, u is identified with the input-signal level. This makes u independent

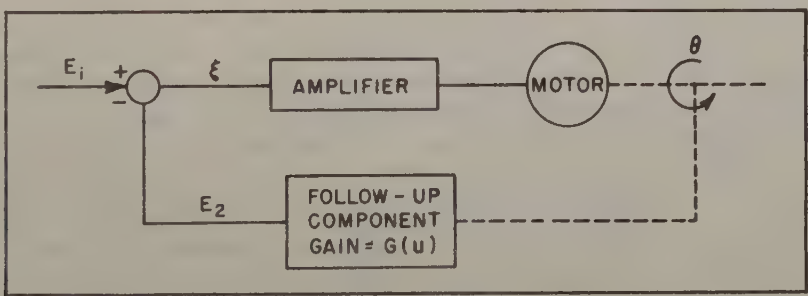

Fig. 1 - Nonlinear servo.

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of the output-shaft position, since the output is dependent upon phase of the input signal but not upon intensity. Cases can be assumed where the variable u is completely divorced from the servo altogether. Since G(u) is a multiplier in the servo loop-gain, a change in this quantity can be expected to cause serious deterioration of the servo performance, independently of how u is related to the servo itself. Under actual operating conditions, variations between the extremes of complete insensitivity and a state of self-oscillation are usual. This paper is intended to show in a practical way how the influence of u on the servo loop can be nullified.

SOURCES OF NONLINEARITY

Consider an amplitude plotter where the output is proportional to a nonlinear function of input and the follow-up component is a potentiometer having fixed excitation. The pertinent example shown in Fig. 2 is a square-rooting device.

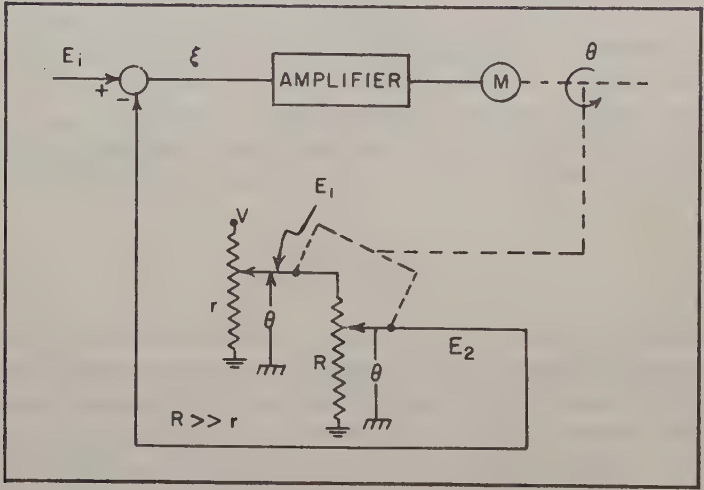


Fig. 2 - Uncompensated square-root servo.

Assuming a perfect servo, the functional relationship $E_2 = k\theta^2$ yields square-root response. Small-signal gain of the follow-up component, G(u), is defined as $dE_2/d\theta$. In this particular instance $G(u) = K\theta$, which shows the parameter u is identified with θ.

Servomechanisms using nonlinear potentiometers in the manner shown may have an incremental gain which is not simply proportional to the slope of the follow-up potentiometer function. Certain other factors must often be taken into consideration in determining $G(u = \theta)$. For instance, the impedance of the input generator may be a contributing factor in the determination of G(u). It is worthy of note that under certain circumstances even a so-called linear servo may exhibit nonuniform incremental loop-gain. It is not within the framework of this paper to consider how the various factors influence G(u) but rather how the servo may be made to have uniform dynamic stability in the presence of such an element. This discussion does not imply that the incremental loop-gain is always a variable in a servo designed to plot a nonlinear function of input-signal amplitude. Consider the particular type of logarithmic recorder shown in Fig. 3.[1,2] Here the input voltage E_i is applied to an exponential attenuator through amplifier K. The pick-off voltage e is servo-positioned for a constant

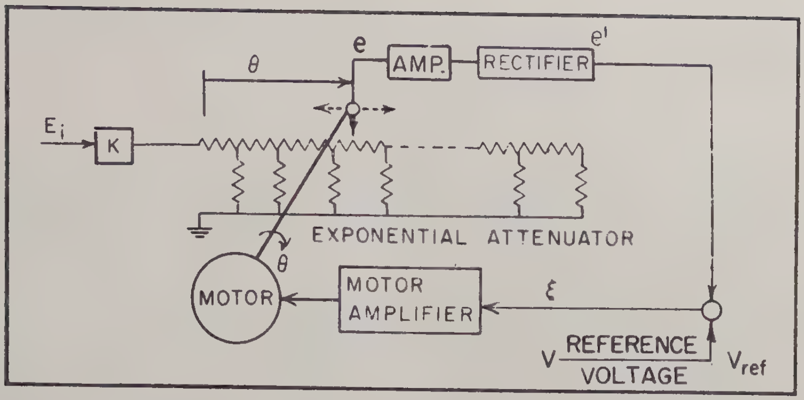


Fig. 3 - Logarithmic servo.

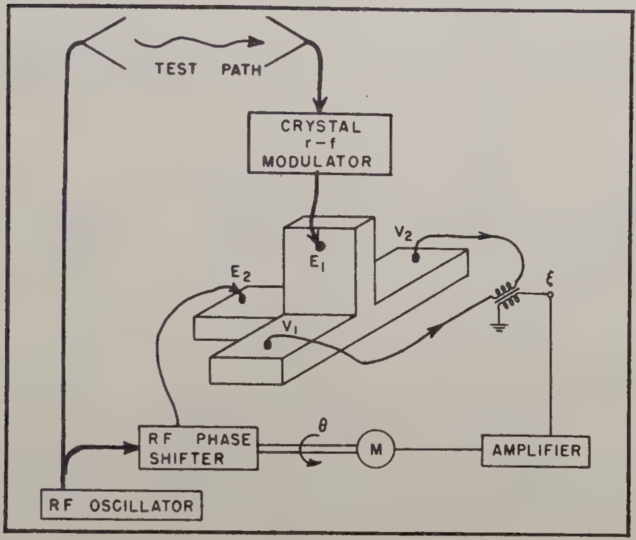


Fig. 4 - Rudimentary phase servo.

signal level so that the rectified voltage e' equals the reference voltage V_{ref}. This makes Eqs. (1) and (2) hold.

$$e = KE_i \epsilon^{-\alpha\theta} = \text{constant} \tag{1}$$

$$\theta = C \log (K' E_i) \tag{2}$$

Symbols C and K' are constants. The incremental gain of the attenuator is given, as before, as the derivative of Eq. (1) with respect to θ. This becomes a new constant. Thus, an exponential attenuator used in this manner behaves for small changes in signal level as if it were linear. The servo performance is consequently independent of input-signal level over the usable dynamic range of the instrument. Recorders constructed using this principle confirm this fact.

There are cases where the independent variable u is identified with the input-signal intensity but not with the output-shaft position. A servo designed to plot rf phase is a particular example.[3,4] In an elementary design shown in Fig. 4, a balanced hybrid junction is employed as the rf-phase discriminator.

Under proper operating conditions the magnitude of the audio error voltage ξ for small values of δ is given by Eq. (3).

$$\xi = KE_1 \, E_2 \delta \tag{3}$$

The angle δ is the amount by which the phase of E_1 and E_2 differs from quadrature. Thus, with a phase difference of $\pi/2$ radians, $\xi = 0$ regardless of the amplitude of E_1. The instrument operates by using ξ to actuate a motor, which in turn moves an rf-phase shifter so as to minimize this error. Thus, as the phase of E_1 varies, the rf-phase shifter "tracks" to maintain E_2 at a constant phase difference of $\pi/2$ radians. The phase-shifter position is taken as an indication of the relative input phase, by actuating a chart mechanism from the mechanical output. Since the rf phase is a linear function of the shaft position, the static sensitivity of the hybrid junction is proportional to $\frac{d}{d\theta} \xi$.

The reference voltage E_2 is normally constant in value, and, consequently, the gain function in this case is proportional to the magnitude of E_1, as is shown in Eq. (4). Thus, u is identified with the input-signal strength and independent of the shaft position.

$$G(u) = \frac{d\xi}{d\theta} = C \frac{d\xi}{d\delta} = K_o E_1 \tag{4}$$

Inasmuch as the magnitude of E_1 may vary widely in actual operation, the utility of the instrument in this simple form is necessarily restricted.

COMPENSATION FOR NONUNIFORM GAIN

Thus far we have discussed how servo loop-gain in some pertinent examples of antenna instrumentation can become a function of an external variable. At this point it would be well to consider some remedial measures which can be taken to maintain uniform transient response over the dynamic range of a servo characterized by including nonlinear elements of the type described.[5] Figure 5

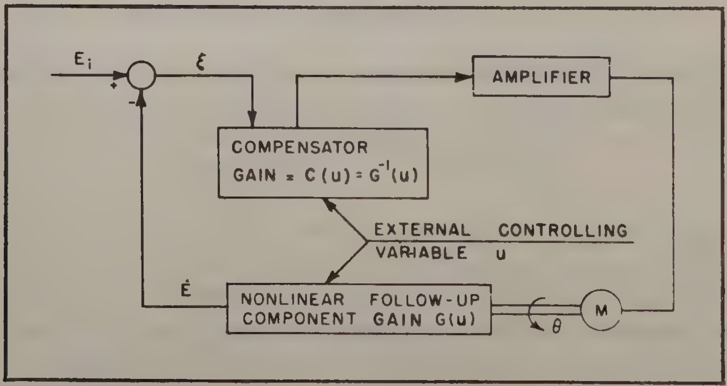


Fig. 5 - Compensated servo with externally controlled nonlinearity.

is a modification of the servo shown in Fig. 1. Under static conditions, where the input is maintained constant, Eq. (5) holds. It is clear in this case that the loop-gain is proportional to the quantity C times G. Thus, for small signals the effect of the variable u on the system

$$\frac{d}{d\theta} E = - \frac{d\xi}{d\theta} = G(u) \tag{5}$$

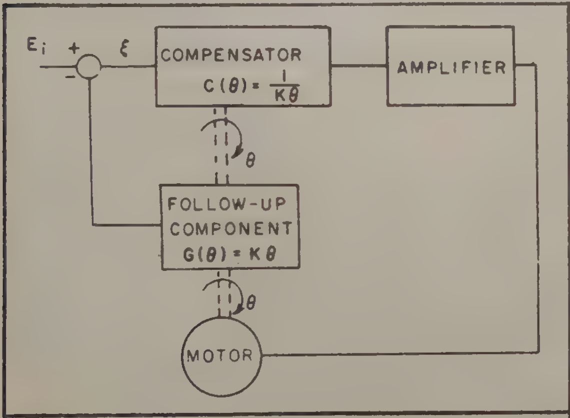
can be nullified if the quantities C and G are reciprocals.

The first of two separate cases to be considered is where the independent variable u is identified with the output-shaft position. The second is where the independent variable u is an external voltage source.

When the variable u is identified with the output-shaft position, the compensator C can be simply an additional potentiometer attached to the output shaft. Consider again the square-root servo of Fig. 2.[6] It was pointed out in this example that $G(u = \theta)$ is proportional to θ. To obtain a reciprocal relationship between C and G, it must be true that $C = K/\theta$. Figure 6 shows the compensator C attached to the output shaft. The reciprocal gain function is approximated by a set of shunts attached at judicious points along the resistance.

It is obvious that a square-rooting servo can never be made to plot precisely to zero, because the follow-up gain $G(\theta)$ becomes vanishingly small in the neighborhood of $\theta = 0$. To maintain constant loop-gain in the neighborhood of the origin would require that the compensator gain increase without bound. Certain compromises can be made in this region, however, without seriously affecting the accuracy of the instrument. For example, modification of the law of response by altering $G(\theta)$ so that it has a finite slope near the origin can be used without causing serious error. Since in this region there is usually noise present which tends to obscure the true reading, a slight distortion of the law of response here has no significant harmful effect. An alternative in this situation is to stop the instrument short of zero. We have done this with instruments constructed at the Ohio State University Antenna Laboratory.

Uniformity of transient response across the dynamic range of the instrument depends upon the accuracy with which the compensator is adjusted. Figure 7 shows the uniformity of response which can be achieved over an 80 db range of input



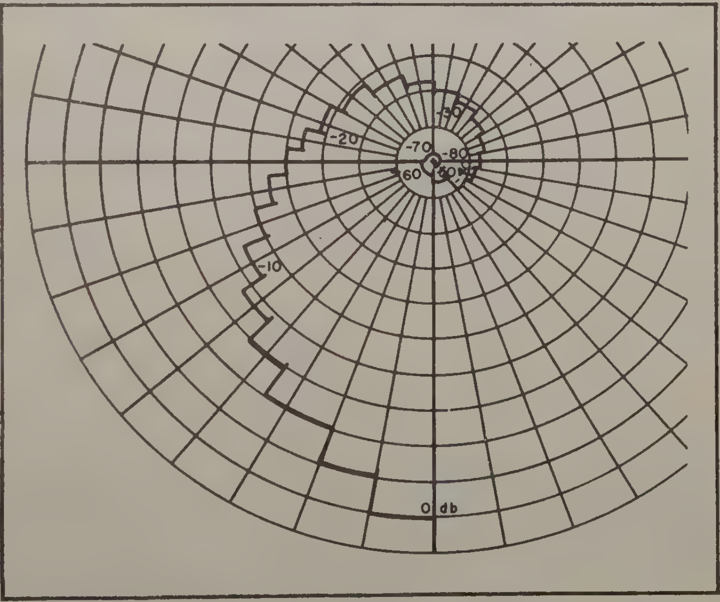


Fig. 6 - (above) Compensated square-root servo.
Fig. 7 - (right) Step function response for a compensated square-root servo.

signal or a 40 db range of output. The response has proved to be sufficiently uniform so that there is little noticeable difference between its behavior and that of a so-called linear servo.

This same technique has also been applied to an instrument servo where the loop-gain varied exponentially with output-shaft position over an amplitude range of 100 db.[7] The compensator consisted of another exponential attenuator attached to the output shaft and having a gain function which varied in an inverse manner to the gain of the follow-up component. The finished design exhibited a uniformity of response very little different from that of a truly linear servo.

In situations where the loop-gain is functionally related to an external variable, say a voltage, the problem of stabilization becomes somewhat more difficult. A practical solution, which places no severe constraints upon changes in the variable u, consists of employing an auxiliary servomechanism to obtain the necessary compensation. A suitable slave potentiometer mounted on the output shaft of the auxiliary servo can provide the required compensation to the principal servo loop. Harmful effects, from the inclusion of a time dependent variable (viz., the compensating potentiometer) into the principal servo loop, can be satisfactorily minimized by making the compensating servo response relatively fast in comparison to that of the principal servo. This property can be readily designed into the system.

The phase plotter shown in Fig. 4 is an example of where an auxiliary servo can effectively provide the required loop-gain stabilization. The necessary

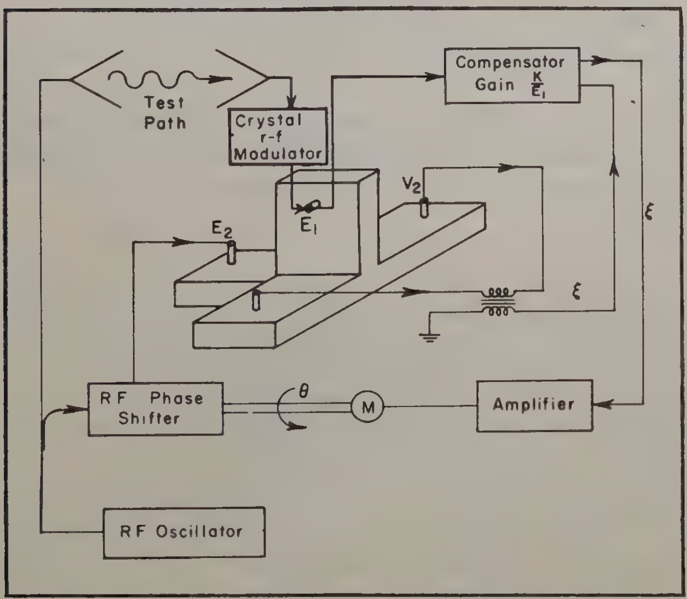


Fig. 8 - Compensated phase servo.

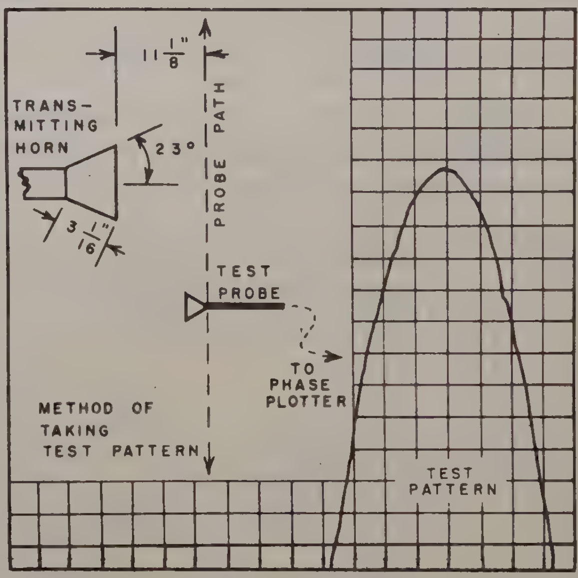


Fig. 9 - Compensated phase servo response.

modifications are shown in Fig. 8. It was previously pointed out that the error comparator has a gain proportional to E_1 but independent of θ. As a result, the compensator-gain function must necessarily be of the form K/E_1 throughout the dynamic range of the instrument. The auxiliary servo is designed to have this property. Figure 9 shows the actual dynamic response of an instrument employing

compensation of the type discussed and having an rf input amplitude free to vary over a 40 db range while the phase is being recorded.[8]

SERVO COMPENSATOR DESIGN

Technical investigations often lead to original by-product results with usefulness extending beyond the initial scope of the inquiry. Such is the case here. The results to be described grew out of a need for a suitable compensating servomechanism for use when the independent variable u was an external voltage. In viewing the desirable properties of a suitable compensator, the following points are salient:

1. It should employ servo techniques.

2. It should be capable of operating with an audio input voltage.

3. It should be capable of operating over a wide dynamic range (80 db of audio).

4. It should be as flexible as possible so that the same basic design may be adapted to a wide variety of compensation requirements.

5. The functional accuracy may be relaxed in comparison to that normally required of an accurate amplitude plotter.

The requirement for a wide dynamic operating range immediately leads us to consider a modified logarithmic servomechanism for use as a compensator. The question which logically arises at this point is: what sort of a slave component will be suitable, and can one form of it fulfill a variety of requirements?

Figure 10 shows a device which has great flexibility in the type of gain function it can simulate and, in addition, has other important features which will be emphasized. The top portion is simply an audio logarithmic servo wherein the follow-up potentiometer is exponential and has an attenuation rate α. The output shaft, having a normalized angle of rotation θ ($0 \leq \theta \leq 1$), is coupled to

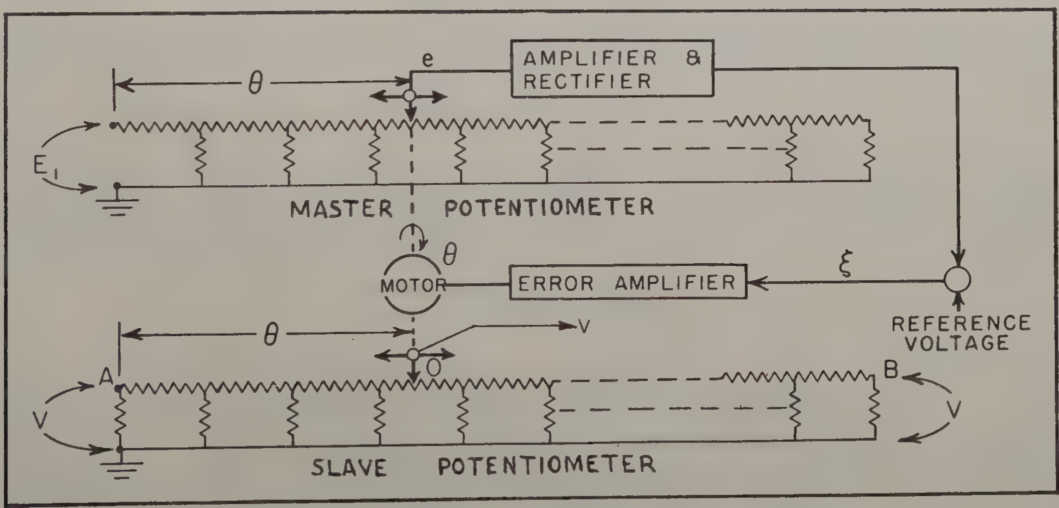


Fig. 10 - Logarithmic servo compensator.

a slave potentiometer. The slave is also an exponentially tapered attenuator but has in general a different rate of attenuation than the master potentiometer. Let us define its attenuation rate as $N\alpha$. The variable u is represented by the voltage E_1 applied to the master potentiometer. The servo positions the pickoff on the potentiometer so as to make its voltage • constant. Thus,

$$e = E_1 \varepsilon^{-\alpha\theta} = \text{constant} \; . \tag{6}$$

The input signal to the compensator is the voltage V which may be applied to either end of the slave potentiometer. Assume that it is first applied to the A end. The output voltage from the compensator is the voltage v taken from the tap O on the slave potentiometer. Since the voltage distribution along this potentiometer is also exponential and has an attenuation rate $N\alpha$, the compensator output v is related to the input V by Eq. (7).

$$v = V\varepsilon^{-N\alpha\theta} \tag{7}$$

The compensator-gain function $C(E_1)$ is the ratio v/V. Combining Eqs. (6) and (7) and rearranging term gives for $C(E_1)$

$$C(E_1) = v/V = KE_1^{-N} \tag{8}$$

where K is a constant. Thus, with this technique we are able to generate a gain function proportional to a voltage E_1 to the minus N power, the N being determined by the ratio of the attenuation rates in the two exponential attenuators. No constraints are placed on N except that it be a positive number.

Now consider a slight modification to the circuit. Transfer the compensator-input V from terminal A on the slave attenuator to terminal B. Equation (9) is valid in this case.

$$v = V\varepsilon^{-N\alpha(1-\theta)} \tag{9}$$

Again substituting from Eq. (6) and reordering the terms yields

$$C(E_1) = v/V = KE_1^{+N} \; . \tag{10}$$

Thus, the method described will lend itself to the generation of gain functions of the form given in Eq. (11).

$$C(u) = Ku^{\pm N} \tag{11}$$

Here u is the generalized control voltage previously discussed in regard to the compensation of loop-gain. It is clear that a wide variety of compensation requirements can be fulfilled using this technique and that more complicated compensation functions could be synthesized by combining the output of several slave components where the exponents in general were different.

The reader will recall that in the phase plotter it was required that the compensator gain be proportional to K/u. This was obtained by constructing both attenuators to have the same attenuation and by using terminal A and the arm respectively as the input and output of the compensator. The logarithmic property of the master servo made it possible to obtain easily an 80 db dynamic range on the input control voltage without any difficulty.

An obvious extension to the described technique is that of using the compensator as a function generator in its own right. This can be done by applying a fixed potential to either terminal A or B of the slave unit and using the pickoff voltage as the output, thereby synthesizing functions of the form given in Eq. (12)

$$v = KE_1^{\pm N} \tag{12}$$

It should be possible, in principle at least, to construct polynominals of the form shown in Eq. (13) if a number of the slave units are employed, each

$$v = \sum_{j = -n}^{j = +m} A_j E^j \tag{13}$$

having a different rate of attenuation. This would, of course, require independent amplifiers to adjust the separate constants and an adder to perform the summation.

A further point worthy of note is the flexibility of the design when used either as a function generator or as a compensator. Recall that in the slave potentiometer the exponent of ε is $N\alpha\theta$. Here N can be associated with either α or θ. By associating N with θ, the law of response can be altered merely by using a gear ratio between the master potentiometer and the slave instead of direct coupling as shown in the figure.

CONCLUSION

In review, we have discussed how certain nonlinearities naturally arise in some important antenna-instrumentation problems. The nonlinearities were divided into two categories. First, there were those which were functions of the output-shaft position; then, those which were dependent upon an external potential, which could be independent of the controlling voltage of the servo. In the first case, it was shown that an additional compensating potentiometer could be affixed to the output shaft and coupled into the error path of the servo to bring about satisfactory gain stabilization, even though the principle was based on small-signal response. In the second case, we saw how an auxiliary compensating servo could be employed successfully to compensate for external influence on the nonlinear servo. This required a technique for obtaining a wide variety of gain functions to meet diversified conditions. The compensator was designed to obtain the desired compensation under a variety of conditions. It was indicated how a slight modification of the compensator would convert it into a function generator for the purpose of generating terms of a constant-coefficient polynominal. There is much work remaining to be done before the full possibilities of the function-generator technique can be effectively assessed. However, at the present time it offers some challenging possibilities.

BIBLIOGRAPHY

1. Tyson, O. A., and Edwards, W. J. E.: "A portable servo recorder for antenna patterns." Naval Research Laboratory Report (July 1949).

2. Bacon, J.: "A 40-db audio logarithmic recorder for antenna measurements." Columbus, Ohio, The Ohio State University Research Foundation, Antenna Laboratory, Report No. 478-10 (April 1953).

3. Hines, J. N.: "An automatic phase plotter." Columbus, Ohio, The Ohio State University Research Foundation, Antenna Laboratory, Report No. 301-31 (April 1951).

4. Bacon, J.: "An automatic X-band phase plotter." Columbus, Ohio, The Ohio State University Research Foundation, Antenna Laboratory, Report No. 531-7 (Sept. 1954).

5. Cosgriff, R.: "A study of nonlinear servomechanisms." Columbus, Ohio, The Ohio State University Research Foundation, Antenna Laboratory, Report No. 478-14 (Aug. 1953).

6. Bacon, J.: "A wide range square root recorder." Columbus, Ohio, The Ohio State University Research Foundation, Antenna Laboratory, Report No. 478-19 (Nov. 1953).

7. Bacon, J.: "A sensitive wide range logarithmic recorder." Columbus, Ohio, The Ohio State University Research Foundation, Antenna Laboratory, Report No. 301-33 (Sept. 1951).

ON THE DESIGN OF A-C NETWORKS FOR SERVO COMPENSATION

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Summary -- This paper presents an analytical method for analysis and synthesis of networks for AC servo compensation.

The response of a linear network to a modulated suppressed-carrier excitation is formulated in terms of in-phase and quadrature carrier components. The relationship between the modulation on these components and the exciting modulation is shown to depend upon operators simply related to the original network function.

The expansion of the data-frequency operators into partial fractions is shown to lead to a simple synthesis procedure for deriving the original network operator from the in-phase or quadrature operators.

As an example, the process is applied to the derivation of a representative network for lead compensation of an AC servo responsive to the in-phase component of error signal.

The use of RC networks in AC servo compensation is shown to be limited to derivative types of equalization.

INTRODUCTION

The carrier-frequency servomechanism occupies an increasingly important place in the instrumentation field. Evidence of this appears continually in advertisements and in papers describing developments. A continuing problem in this field is in improving methods for analyzing and selecting networks in order to stabilize or compensate carrier servos without resorting to demodulation techniques. Starting with Sobczyk's work in 1945, the emphasis has been on frequency analysis techniques, and methods of approximation based upon frequency response.[1, 2] Probably the most popular and effective method to date has been the use of low-pass to band-pass network transformations as a means of finding appropriate networks.[3]

In this paper we take a different approach. Using Laplace transform techniques we first develop exact relations for network response to suppressed carrier modulation. The resulting relationship is examined in terms of the pole-residue parameters of the network. From this we develop specifications that a required modulation operator must meet in order to be realizable. Using the pole-residue relationships, we develop a method for modifying a given operator in order to make it realizable; a technique for recovering the realizable data frequency operator is then described.

The results are applied to the analysis of the general performance of RC networks and to derivation of a network transfer function providing lead-lag operation on the modulation.

The work assumes that a servo motor is equivalent to a demodulator responsive to the modulation on the in-phase component of the carrier.

THE DATA FREQUENCY TRANSFER FUNCTIONS

It is assumed that the input signal is a suppressed carrier modulated signal, as commonly encountered in AC servos. It may arise from the error signal of a synchro control transformer or as the consequence of mixing command voltage and feedback voltage. The essential feature is that it may be represented mathematically as

$$e_s = e(t) \cos w_c t \tag{1}$$

The situation is depicted in Figure 1.

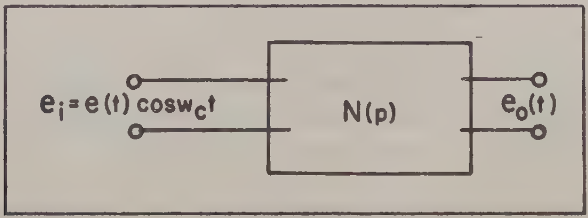

Fig. 1 - Diagram of problem.

In Appendix A we derive the following result for the output of the network:

$$e_o = \cos w_c t \left[\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} du \; E(u) \; \exp(ut) \left\{ \frac{1}{2} \left(N(u + iw_c) + N(u - iw_c) \right) \right\} \right] \tag{2}$$

$$+ \sin w_c t \left[\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} du \; E(u) \; \exp(ut) \left\{ \frac{i}{2} \left(N(u + iw_c) - N(u - iw_c) \right) \right\} \right]$$

In this equation $E(u)$ is the Laplace transform of $e(t)$, and $N(u + iw_c)$ and $N(u - iw_c)$ are the transfer functions of the network evaluated at $p = (u + iw_c)$ and $p = (u - iw_c)$. It appears from this expression that the output of the network can be regarded as the sum of two independently modulated suppressed-carrier components, one in-phase with the original carrier, and the other, the sin term, in quadrature with the carrier. The modulation on each component is seen to be the inverse Laplace transform of $E(u)$ multiplied by a suitable function of u that depends on the network. We may regard these functions as the data-frequency operators of the network. An equivalent circuit is shown in Figure 2.

This formulation is quite general. It has not been necessary to specify that the response is a current or a voltage, so that N may represent a driving point impedance or admittance, or a transfer function. It merely states that if the system function relating two variables in a linear network is known, and

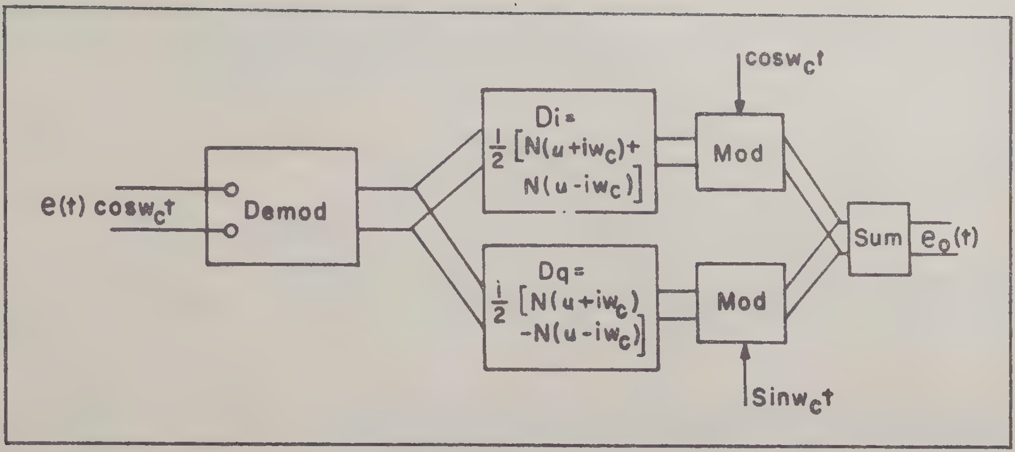

Fig. 2 - Equivalent data frequency operation.

if one of the variables is suppressed-carrier modulated, the other consists of two suppressed-carrier modulated components, whose modulation is uniquely related to the reference modulation through the pair of operators. For convenience we define the operators as

$$D_i = \frac{1}{2}\left[N(u+iw_c)+N(u-iw_c)\right] \tag{3}$$

$$D_q = \frac{i}{2}\left[N(u+iw_c) -N(u-iw_c)\right] \tag{4}$$

where D_i and D_q are the in-phase and quadrature operators respectively.

ADDITION AND MULTIPLICATION PROPERTIES OF THE OPERATORS

If two networks are connected so that their responses to an input signal are added to form the total response, the data-frequency operators add to form the net data-frequency operators.

$$\left.\begin{aligned} D_{i1}+D_{i2} &= D_i \\ D_{q1}+D_{q2} &= D_q \end{aligned}\right\} \tag{5}$$

However, if two networks are connected in cascade, their data-frequency operators do not multiply, as do the original network system functions. The overall system function must be calculated in order to determine the overall data-frequency functions.

$$\begin{aligned} D_i &= \frac{1}{2}\left[N_1(u+iw_c)\,N_2(u+iw_c)+N_1(u-iw_c)\,N_2(u-iw_c)\right] \\ D_q &= \frac{i}{2}\left[N_1(u+iw_c)\,N_2(u+iw_c)-N_1(u-iw_c)\,N_2(u-iw_c)\right] \end{aligned} \tag{6}$$

Fortunately, the addition and multiplication operations are commutative. That is, the order in which the networks are added, or the order in which they are cascaded, do not alter the overall data-frequency operators.

It will appear later that these properties dictate the manner in which the network properties are synthesized.

PROPERTIES OF THE DATA-FREQUENCY TRANSFER FUNCTION

The data-frequency operators are the sum and difference of $N(u+iw_c)$ and $N(u - iw_c)$.

These functions are complex rational functions of the complex variable u. The original transfer function N is a real rational function of p. It may be written as

$$N(p) = \frac{P(p)}{Q(p)} \tag{7}$$

where P and Q are real polynomials in p.

The roots of Q(p) (the poles of N) must lie in the left half of the p plane. Real roots lie on the negative p axis; complex roots occur in conjugate pairs.

The roots of P may lie anywhere in the p plane, with the proviso that complex roots occur in conjugate pairs.

The degree of the polynomial P must be equal to or less than the degree of the polynomial Q. There is also a constant multiplier associated with N. The maximum permissible value depends upon the network topology by which N is to be realized. For the purposes of this analysis we may ignore the multiplier. However, in the determination of network component values, it would be necessary to consider the multiplier.

Under the transformations $p = u + iw_c$ and $p = u - iw_c$, the poles and zeros of N are shifted down and up respectively in the complex u plane. This situation is shown in Figure 3. It is obvious that for every root of $N(u+iw_c)$ there is a corresponding conjugate root in $N(u - iw_c)$.

If the zeros of N are denoted by Greek subscripts and the poles by Latin subscripts, we have the following relations:

$$N(p) = \frac{(p - p_\alpha)(p - p_\beta)\ \cdots\cdot}{(p - p_a)(p - p_b)\ \cdots\cdot} \tag{8}$$

$$D_i = \frac{1}{2}\left\{\frac{(u - p_\alpha + iw_c)(u - p_\beta + iw_c)\cdots\cdot}{(u - p_a + iw_c)(u - p_b + iw_c)\cdots\cdot} + \frac{(u - p_\alpha - iw_c)(u - p_\beta - iw_c)\ \cdots\cdot}{(u - p_a - iw_c)(u - p_b - iw_c)\ \cdots\cdot}\right\} \tag{9}$$

$$D_q = \frac{i}{2}\left\{\frac{(u - p_\alpha + iw_c)(u - p_\beta + iw_c)\ \cdots\cdot}{(u - p_a + iw_c)(u - p_b + iw_c)\ \cdots\cdot} - \frac{(u - p_\alpha + iw_c)(u - p_\beta + iw_c)\ \cdots\cdot}{(u - p_a - iw_c)(u - p_b - iw_c)\ \cdots\cdot}\right\} \tag{10}$$

There are three distinguishable root locations. If a root of N is on the real axis, $p = +a$, it appears as a complex root $u = a - iw_c$ in $N(u + iw_c)$ and the conjugate $u = a + iw_c$ in $N(u - iw_c)$.

If a general pair of complex conjugate roots $p = (a \pm ib)$ appears in N, it transforms into $a + i(b - w_c)$ and $a - i(b + w_c)$ in $N(u - iw_c)$, and $a + i(b + w_c)$ and $a - i(b - w_c)$ in $N(u + iw_c)$.

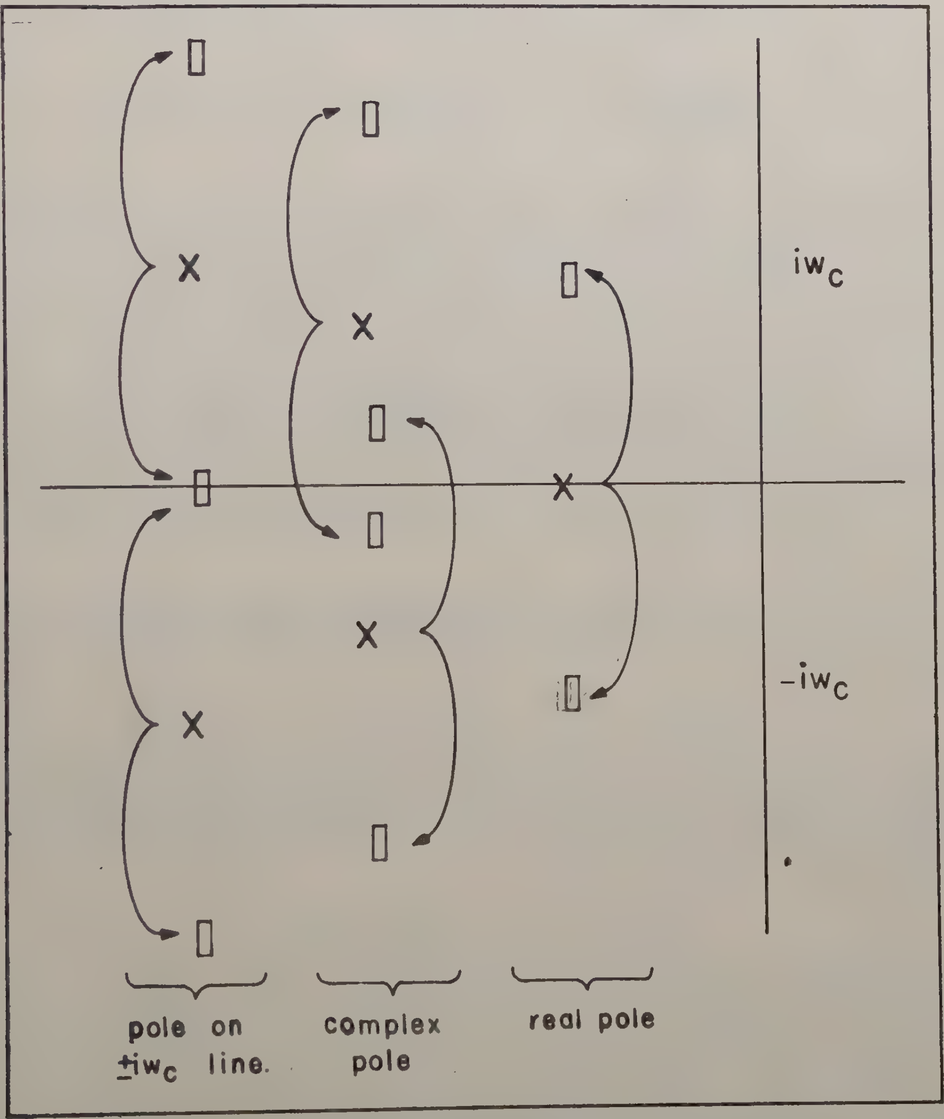

Fig. 3 - Movement of typical poles of N.

Finally, if the complex pair takes the form $p=(a \pm iw_c)$, they transform into a, a−2 iw_c and a, a+2 iw_c. That is, both N functions contain the factor (u − a).

Attention will now be concentrated on the D_i function.

$$D_i = \frac{1}{2}\left[N(u + iw_c) + N(u - iw_c)\right]$$

$$= \frac{1}{2}\left[\frac{P(u + iw_c)}{Q(u + iw_c)} + \frac{P(u - iw_c)}{Q(u - iw_c)}\right]$$

$$D_i = \frac{1}{2}\left[\frac{P(u + iw_c)\ \ Q(u - iw_c) + P(u - iw_c)\ Q(u + iw_c)}{Q(u + iw_c)\ \ Q(u - iw_c)}\right] \quad (11)$$

$$= \frac{P_i}{Q_i}$$

Q_i contains all the roots of Q repeated twice, once shifted up iw_c and once shifted down iw_c. Complex roots occur in conjugate pairs. Therefore, Q_i is a real function of u. Since the real parts of its roots have not been altered, all of the roots lie in the left half u plane. If a root transforms into a real root it appears once in P_i and twice in Q_i. Thus, the degree of the polynomial Q_i is equal to or less than twice the degree of Q. It is exactly equal to twice the degree of Q less the number of roots of Q having an imaginary part equal to w_c. A real root in Q transforms into a complex conjugate pair. A complex conjugate pair transforms into a pair of complex conjugate pairs, except for the special case when it transforms into a triple consisting of a real root and a complex conjugate pair.

The roots of P_i are not readily established. P_i is the sum of two polynomials whose coefficients are complex conjugates of each other, so that P_i is a real function of u.

If a root of P transforms into a real root, this becomes a root of P_i and the corresponding term may be factored out. However, the roots may lie anywhere in the complex u plane without any simple relationship to the roots of P_i. The degree of the polynomial P_i is equal to or less than the sum of the degrees of P and Q on reduction to simplest form.

PARTIAL FRACTION EXPANSION

In determining the desired data-frequency transfer function for a servo, it is customary to express it as a ratio of numerator factors divided by denominator factors. That is, ordinarily the poles and zeros of the desired operator are specified. While this determines a unique function, it is not the only way to express the operator mathematically. This is fortunate, for as was noted above the zeros of the data-frequency operators are not readily

ascertained. For this purpose the partial fraction expansion is most useful. Instead of expressing N as a ratio of two polynomials, it proves desirable to express it in partial fractions.

$$N = \frac{P}{Q} = a_o + \frac{a_1}{p - p_1} + \frac{a_2}{p - p_2} + \frac{a_3}{p - p_3} + \dots \tag{13}$$

In this form the P_i are the poles of N, a_o is the residue at infinity and may be zero (when the degree of P is less than the degree of Q), and the a_k are the residues at the poles of N. The a_k generally are complex, occuring in conjugate pairs associated with the complex conjugate poles. The real parts of the a_k may be negative.

Forming the expression for D_i in these terms

$$N(u + iw_c) = a_o + \frac{a_1}{u - p_1 + iw_c} + \frac{a_2}{u - p_2 + iw_c} + \dots \tag{14}$$

$$N(u - iw_c) = a_o + \frac{a_1}{u - p_1 - iw_c} + \frac{a_2}{u - p_2 - iw_c} + \dots \tag{15}$$

$$D_i = a_o + \frac{1}{2}\left(\frac{a_1}{u - p_1 + iw_c} + \frac{a_1}{u - p_1 - iw_c}\right) + \frac{1}{2}\left(\dots\right. \tag{16}$$

If p_i is real, then the typical term d

$$d = \frac{1}{2}\left(\frac{a_1}{u - p_1 + iw_c} + \frac{a_1}{u - p_1 - iw_c}\right)$$

consists of a pair of complex conjugate poles.

If p_i is complex its conjugate will also be found and the group of terms associated with it becomes

$$d = \frac{1}{2}\left[\frac{a_1}{u - p_1 + iw_c} + \frac{a_1^*}{u - p_1^* - iw_c} + \frac{a_1^*}{u - p_1 - iw_c} + \frac{a_1}{u - p_1^* + iw_c}\right] \tag{17}$$

This shows that in the D_i function every complex pole is accompanied by a complex conjugate pole and the residues at these poles are also complex conjugate, derivable directly from the corresponding residues of N.

Thus, the essential transformation properties of D_i are these: the poles of N are carried over in duplicate (as described earlier) and the residues at the poles are carried over intact. The problem of inverting the transformation can be accomplished by means of them.

THE INVERSION PROCESS

The process of inversion, of finding N from D_i, is primarily one of identifying poles and residues with the poles and residues of $N(u+iw_c)$ and $N(u-iw_c)$ and then shifting the poles as required. The rules are relatively simple.

1. Express D_i in partial fractions. In the equation below, real and imaginary parts of the poles and residues are explicitly denoted in order to indicate the difference between real and complex poles. If D_i is a proper function, it will take the following most general form:

$$D_i = a_0 + \underbrace{\frac{a_1}{u-\alpha_1+iw_c} + \frac{a_1}{u-\alpha_1-iw_c}}_{\text{terms corresponding to real roots in N.}} + \dots \tag{18}$$

$$+ \underbrace{\frac{a_2+ib_2}{u-\alpha_2+2\,iw_c} + \frac{2a_2}{u-\alpha_2} + \frac{a_2-ib_2}{u-\alpha_2-2\,iw_c}}_{\text{terms corresponding to complex pairs } (\alpha_2 \pm iw_c)} + \dots$$

$$+ \underbrace{\frac{a_3+ib_3}{u-\alpha_3+i\beta_3} + \frac{a_3-ib_3}{u-\alpha_3-i\beta_3} + \frac{a_3-ib_3}{u-\alpha_3+i\beta_4} + \frac{a_3+ib_3}{u-\alpha_3-i\beta_4}}_{\text{terms corresponding to general complex pairs}}$$

$$\left(\alpha_3 \pm i\,\frac{\beta_3+\beta_4}{2}\right) \quad \text{with} \quad \frac{\beta_3-\beta_4}{2} = \pm\, w_c$$

2. From this group identify the partial fraction expansion of $N(u+iw_c)$ as follows:

$$N(u+iw_c) = a_0 + \frac{2a_1}{u-\alpha_1+iw_c} + \dots \tag{19}$$

$$+ \frac{2(a_2+ib_2)}{u-\alpha_2+2iw_c} + \frac{2(a_2-ib_2)}{u-\alpha_2} + \dots$$

$$+\frac{2(a_3+ib_3)}{u-\alpha_3+i\beta_3}+\frac{2(a_3-ib_3)}{u-\alpha_3-i\beta_4}+\ldots$$

3. Make the substitution $u=(p-iw_c)$. Then $N_{(p)}$ is given by

$$N_{(p)}=a_0+\frac{2a_1}{p-\alpha_1}+\ldots \tag{20}$$

$$+\frac{2(a_2+ib_2)}{p-\alpha_2+iw_c}+\frac{2(a_2-ib_2)}{p-\alpha_2-iw_c}+\ldots$$

$$+\frac{2(a_3+ib_3)}{p-\alpha_3+i(\beta_3-w_c)}+\frac{2(a_3-ib_3)}{p-\alpha_3-i(\beta_4+w_c)}$$

THE QUADRATURE OPERATOR

The general properties of the quadrature operator are the same as those of the in-phase operator, in terms of poles and zeros. However, due to the i multiplier and the negative sign associated with $N(u-iw_c)$, the emphasis in the partial fraction expansion is on the imaginary part of the residues. The typical form for D_q corresponding to equation 18 is displayed in equation 21.

$$D_q=\frac{ia_1}{u-\alpha_1+iw_c}\quad\frac{ia_1}{u-\alpha_1-iw_c}+\ldots \tag{21}$$

$$-\frac{b_2-ia_2}{u-\alpha_2+2iw_c}+\frac{2b_2}{u-\alpha_2}-\frac{b_2+ia_2}{u-\alpha_2-2iw_c}+\ldots$$

$$-\frac{b_3-ia_3}{u-\alpha_3+i\beta_3}-\frac{b_3+ia_3}{u-\alpha_3-i\beta_3}+\frac{b_3+ia_3}{u-\alpha_3+i\beta_4}+\frac{b_3-ia_3}{u-\alpha_3-i\beta_4}+\ldots$$

Note that the constant term has disappeared and that the residue associated with the real poles is the imaginary part of the residue in N. The inversion process can be performed through equation 21. The constant part of N is then arbitrary.

AN EXAMPLE

An example may serve to clarify some of the problems and illustrate the technique of inversion.

Consider the following problem. The power phase of a two-phase servo motor is connected so that the motor is responsive to the in-phase carrier component. This is a common situation. It has been determined that the transfer function operating on the data should approximate that of the ordinary lead-lag network. That is

$$D_i \cong \frac{u+a}{u+b} \quad \text{where } a < b \tag{22}$$

We append to this the requirement that D_q should be small at modulation frequencies. This is desirable since the quadrature component acts to reduce the dynamic range of the amplifiers and to heat the motor without producing torque.

Let

$$\tilde{D}_i = \frac{u+a}{u+b} \quad ; \quad a < b \tag{23}$$

$$\tilde{D}_i = 1 - \frac{b-a}{u+b} \tag{24}$$

It is clear from the previous discussion that this does not have the proper form, since every real pole must be accompanied by a pair of complex conjugate poles. Therefore, we modify $\tilde{D}_i$ by adding to it additional terms.

$$\tilde{D}_i = 1 - \frac{b-a}{u+b} - \frac{1}{2}\left[\frac{(b-a)+i\beta}{u+b+2iw_c}\right] - \frac{1}{2}\left[\frac{(b-a)-i\beta}{u+b-2\,iw_c}\right] \tag{25}$$

β is arbitrary. We may use it as a means of adjusting D_q.

We can make a further adjustment to compensate for the introduction of the two complex poles. In the vicinity of $u=0$ (the spectral range of servo modulation) the two complex poles behave like constants. They are ordinarily located so far away from zero that we can add to the constant term an additional term designed to eliminate this zero frequency behavior and expect this situation to hold over a wide range. In this case, the constant would be

$$1/2 \left[\frac{2b\,(b-a) + 4\,w_c\beta}{4\,w_c^2 + b^2}\right]$$

Since our technique is not affected by this, the modification will not be introduced here.

$\tilde{D}_i$ is now in a suitable form for the inversion.

$$N(u+iw_c) = 1 - \left[\frac{(b-a)+i\beta}{u+b+2iw_c} + \frac{(b-a)-i\beta}{u+b} \right] \tag{26}$$

$$N(p) = 1 - \left[\frac{(b-a)+i\beta}{p+b+iw_c} + \frac{(b-a)-i\beta}{p+b-iw_c} \right] \tag{27}$$

Calculate D_q.

$$D_q = \frac{i}{2} \left[N(u+iw_c) - N(u-iw_c) \right] \tag{28}$$

$$= \frac{i}{2} \left[\frac{-4iw_c\,(b-a) + i2\beta\,(u+b)}{(u+b)^2 + 4\,w_c^2} - \frac{2\,i\beta}{u+b} \right] \tag{29}$$

$$= \frac{2\,w_c\,(b-a) - \beta\,(u+b)}{(u+b)^2 + 4\,w_c^2} + \frac{2\beta}{u+b} \tag{30}$$

The selection of β as zero reduces D_q to

$$D_q = \frac{2\,w_c\,(b-a)}{(u+b)^2 + 4\,w_c^2} \tag{31}$$

$$\cong \frac{b-a}{2\,w_c} \quad \text{for} \quad u \cong 0. \tag{32}$$

For this selection of β, the modulation on the quadrature component is proportional to the input modulation, to $(b-a)$ and inversely proportional to the carrier frequency.

From (27) with $\beta = 0$

$$N(p) = 1 - \frac{(b-a)}{p+b+iw_c} - \frac{(b-a)}{p+b-iw_c} \tag{33}$$

$$N(p) = \frac{p^2 + 2\,a\,p + (w_c^2 + 2\,ab\, - b^2)}{(p+b)^2 + w_c^2} \qquad (34)$$

USE OF RC NETWORKS

It is sometimes a self-imposed requirement of the designer that he limit his selection of components for networks to resistors and capacitors. The data-frequency operators of RC networks necessarily have their poles distributed along the $\pm$ jwc lines in the u plane where, as poles, they contribute little to the operations on the modulation spectrum. However, under these circumstances it is necessary to scrutinize the location of the zeros of the data-frequency operators, since by suitable selection of residues it is possible to make them the effective part of the overall operator. A case in point is the parallel Tee network often used to provide data-frequency lead. The transfer function of the network can be taken as:

$$N = \frac{p^2 + w_c^2}{p^2 + 2\,a\,w_c + p + w_c^2} \qquad (35)$$

The numerator has purely imaginary roots at $\pm\, iw_c$. As required in an RC circuit the denominator has real roots, equal when a is unity. The frequency-shifting transformations carry N into the two conjugate functions $N(u + iw_c)$ and $N(u - iw_c)$.

$$N(u + iw_c) = \frac{u\,(u + 2\,iw_c)}{\left[u + w_c\,(a + \sqrt{a^2 - 1} + i)\right]\left[u + w_c\,(a - \sqrt{a^2 - 1} + i)\right]}$$

$$N(u - iw_c) = \frac{u\,(u - 2\,iw_c)}{\left[u + w_c\,(a + \sqrt{a^2 - 1} - i)\right]\left[u + w_c\,(a - \sqrt{a^2 - 1} - i)\right]}$$

$$D_i = u\,\frac{u^3 + 2\,aw_c\,u^2 + 4\,w_c^2\,u + 4\,a\,w_c^3}{u^4 + 4aw_c\,u^3 + u^2(4\,a^2\,w_c^2 + w_c^2) + 8\,a\,w_c^3\,u + 4\,a^2\,w_c^4} \qquad (36)$$

$$\cong \frac{1}{a\,w_c}\,u \text{ if } w_c \gg u \qquad (37)$$

A better approximation in the low-frequency range is

$$D_i \cong \frac{1}{a\,w_c}\left(\frac{1}{1+\frac{2u}{aw_c}}\right) u; \quad w_c > u \tag{38}$$

This indicates the presence of low-frequency lag. Note that the scale factor associated with the differentiation operation is inversely proportional to the carrier frequency. This is not the case with the network designed previously by way of example. There (equation 23), with a taken as zero, the scale factor is approximately $1/b$. Since there is no point in controlling this break frequency as far out as $b = w_c$, it is possible to obtain higher gains with this circuit. Note also that the selection of β relatively small compared with w_c keeps the quadrature noise down.

The quadrature function for the parallel Tee is

$$D_q = \frac{-4\,a\,w_c^2\,u^2}{u^4 + 4aw_c\,u^3 + u^2\,(4\,a^2\,w_c^2 + w_c^2) + 8\,a\,w_c^3\,u + 4\,a^2\,w_c^4} \tag{39}$$

Thus, the low-frequency behavior of D_q is approximated by:

$$D_q \cong -\frac{1}{a\,w_c^2}\,u^2 \tag{40}$$

LOW-FREQUENCY LAG CONTRIBUTION OF RC NETS

In equation (38) it was observed that the denominator of the RC network data-frequency operator could be fairly well approximated by a simple first-degree function of u, contributing a lag term at low frequencies. Since all of the roots of the denominator are on the $+jw_c$ or $-jw_c$ lines this is a fairly good approximation. The time constant of this lag is the sum of the time constants of the denominator first-degree factors, that is, the sum of the reciprocals of the roots of the denominator.

We have then, T, the approximate time constant is

$$T = -\sum \frac{1}{x_k} \tag{41}$$

Since the roots x_k are the real poles of N shifted $\pm iw_c$ we have

$$T = \sum \frac{2\alpha_i}{\alpha_i^2 + w_c^2} \tag{42}$$

where poles of N are given by $(-\alpha_i)$.

Thus, in the parallel T we have poles at $p = w_c\,(-a \pm \sqrt{a^2 - 1})$.

Substituted in (42) these lead to a time constant

$$T = \frac{2}{aw_c}$$

Generally, if the designer does not require control over too wide a bandwidth, he can employ RC networks to generate any degree of derivative type compensation, accompanied by a small, simple lag term.

EFFECTS OF SHIFTING MOTOR REFERENCE PHASE

If the motor reference phase is shifted through an angle θ so that it is no longer wholly in quadrature with the signal reference carrier, the useful data-frequency operator is a linear combination of D_i and D_q. If the motor is responsive to the cos $(wt + \theta)$ component of the carrier the effective data operator is

$$D = \cos\theta\, D_i - \sin\theta\, D_q.$$

The effective quadrature operator is

$$D = \sin\theta D_i + \cos\theta\, D_q.$$

The properties of these functions are generally the same as those of D_i. Since maintaining a specified phase shift different from zero or ninety degrees is usually difficult, it appears unlikely that much use can be made of this relationship.

CONCLUSION

It has been shown that the data-frequency operators of a network excited by a suppressed-carrier modulated signal are readily derivable from the network parameters. When a specification exists for such an operator, it is readily checked for realizability and various artifices may be employed to modify it into appropriate form. Once in suitable form the process of converting the data-frequency operator into the basic carrier-frequency operator is accomplished by simple algebraic means. There are usually at hand a sufficient number of arbitrary parameters to permit simultaneous minimization of the quadrature signal component.

No special reference has been made to detailed methods of realizing the network transfer function as a network, since the literature has made many methods available, and the designer usually has his own preferred technique. (Reference 3)

The use of RC networks is shown to be directed primarily toward derivative types of compensation. If control of lag operators is also desired,

it is necessary to resort to circuits possessing lightly damped poles, i.e., RLC circuits of high Q components.

By means of the techniques described, it is expected that the variety of networks useful for AC servo compensation can be extended beyond those that have been exploited on empirical grounds.

APPENDIX -- DERIVATION OF DATA-FREQUENCY OPERATORS

Given a signal e_s of the form

$$e_s = e(t) \cos w_c t.$$

Assume it applied to a network whose system function is H(p). That is

$$R(p) = H(p) \cdot E(p)$$

Where R(p) is the response, E(p) is the excitation, expressed as Laplace transforms. Then

$$R(p) = H(p)\ L(e_s)$$

$$L(e_s) = \int_0^\infty e(t) \cos w_c(t) \exp(-pt)dt$$

$$= \frac{1}{2} \int_0^\infty e(t) \left[\exp(-p - i\, w_c)t\right] dt$$

$$+ \frac{1}{2} \int_0^\infty e(t) \left[\exp(-p + i\, w_c)t\right] dt$$

$$= \frac{1}{2} \left[E(p + iw_c) + E(p - iw_c)\right]$$

$$R(p) = \frac{1}{2} \left[E(p + iw_c)\ H(p) + E(p - iw_c)\ H(p)\right]$$

Take the inverse Laplace transform

$$r(t) = \frac{1}{2} \left[\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} E(p + iw_c)\ H(p) \exp(pt)\ dp\right.$$

$$\left. + \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} E(p - iw_c)\ H(p) \exp(pt)\ dp\right]$$

In the first term above, let $p = u - iw_c$; in the second let $p = v + iw_c$.

$$r(t)=\frac{1}{2}\left\{\frac{1}{2\pi i}\left[\int_{-i\infty}^{i\infty} E(u)\, H(u - iw_c)\, \exp\,(u - iw_c)t\; du\right.\right.$$

$$\left.\left.+\int_{-i\infty}^{i\infty} E(v)\, H(v + iw_c)\, \exp\,(v + iw_c)t\; dv\right]\right\}$$

Since the integrals are each definite, over the infinite range we may set v equal to u without altering the values of the integrals. Combining

$$r(t) = \frac{1}{2}\left\{\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} E(u) \begin{bmatrix} H(u - iw_c)\, \exp(u - iw_c)t \\ + H(u + iw_c)\, \exp(u + iw_c)t \end{bmatrix} du\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} du\, E(u)\, \exp(ut)\left[H(u - iw_c)\, \exp(-iw_c t) + H(u + iw_c)\, \exp(+iw_c t)\right]\right\}$$

Let $\exp\,(iw_c t) = \cos w_c t + i \sin w_c t$

$\exp\,(-iw_c t) = \cos w_c t - i \sin w_c t$

Combine and factor out the cosine and sine terms.

$$r(t) = \cos w_c t\left\{\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} du\, E(u)\, \exp(ut).\quad \frac{1}{2}\left[H(u + iw_c) + H(u - iw_c)\right]\right\}$$

$$+ \sin w_c t\left\{\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} du\, E(u)\, \exp(ut).\quad \frac{i}{2}\left[H(u + iw_c) - H(u - iw_c)\right]\right\}$$

For the transfer function of a network we have used the symbol N as a special case of H.

REFERENCES

1 - A. Sobczyk, "Stabilization of Carrier Frequency Servomechanisms;" Journal of the Franklin Institute, Vol. 246, 1948, Part I - pp 21-43, Part II - pp 95-121.

2 - H. Hellerman, "Transfer Functions for Amplitude-Modulated Signals;" AIEE Communication and Electronics, No. 22, January 1956, pp 729-731.

3 - J.G. Truxal, Control System Synthesis, McGraw-Hill Publishing Co.,1955.

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FUNDAMENTAL EQUATIONS FOR THE APPLICATION OF STATISTICAL TECHNIQUES TO FEEDBACK-CONTROL SYSTEMS*

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Summary -- The basic equations necessary for applying statistical techniques to the design of feedback-control systems are presented. The autocorrelation function of the output is computed by a transient technique which treats the input autocorrelation function as a transient input to the system. By transforming this procedure equations are developed for relating the spectral densities of the input and output, and a means of performing the computation on an analog computer is presented.

INTRODUCTION

Scope

The basic mathematical equations necessary for the application of statistical techniques to the design of feedback control systems are presented in a concise, yet reasonably simple manner. This paper gives a simple physical interpretation of the statistical calculations by showing that the various correlation functions can be related to each other by transient techniques.

The analysis to be described contains a number of important theoretical gaps and hence is not mathematically rigorous. However, the purpose of the analysis is merely to develop the statistical tools necessary for engineering computation and to present a plausibility argument to give the engineer a physical understanding of the concepts involved. The basic material of this paper is in no sense original. The author is presenting well-known statistical relations in a manner which, he believes, makes them more understandable and useful to the engineer. For more detailed information the reader is referred to Truxal[4] and James, Nichols and Phillips.[5]

This paper shows how to calculate the mean-square value of the response of a linear system from the autocorrelation function of the input and the system transfer function (or impulse response). It presents two methods for performing this calculation, a transient and a transform method. The transient method treats the autocorrelation function of the input as a transient input to the system and calculates the mean-square value of the output by determining the system response to that transient input. The transform method is derived by taking the transforms of the operations involved in the transient method.

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Difference Between the Analysis of Noise and Signal

Statistical techniques are very useful for studying the effect of noise in the reference input to a feedback control system. A detailed knowledge of the noise response is generally not necessary; usually a statistical measure is adequate. In fact, when a designer actually has a complete plot of the response to a noise input, he often condenses the data statistically in order to define it quantitatively. Consequently, it is often desirable to perform in a statistical manner the complete analysis of the effect of the noise by condensing the noise input into a convenient statistical form and from this compute the necessary statistical properties of the output. The statistical condensation of the input that is used is the input autocorrelation function $\Phi_{ii}(\tau)$. Since this is much simpler in form than the actual input $x_i(t)$, it has much less information but does contain sufficient information for determining the mean-square value of the response of any linear system to that input.

Although statistical analysis is usually adequate for determining the effect of a noise input to a feedback control system, it is generally not adequate for determining the effect of the signal portion of the reference input, which the system is designed to follow. One is usually interested in much more information about the system response to the signal portion than merely the mean-square error. Besides, the signal portion often cannot be considered statistically stationary, and a given piece of signal input data is often of such low frequency with respect to the period over which it is measured that a statistically meaningful autocorrelation function of the signal portion cannot be obtained. If an autocorrelation function is computed from the data available, it often does not yield reasonably accurate information concerning the mean-square error of the system response. On the other hand, a plot of the signal portion of the reference input is generally fairly simple in form because the signal portion must be of low frequency with respect to the bandwidth of the system in order for the system to follow it. Consequently, there is no real need to condense the input information statistically before calculating its response.

Thus, for feedback-control applications the statistical techniques should be used in general only for studying the response to noise inputs. When studying the signal portion of the reference input, the complete response can be calculated quite readily by the techniques mentioned in an earlier paper which presents a graphical method for calculating the response of a system to an arbitrary input by expressing the response as a sum of error coefficient terms and transient terms.[1] In fact, this method can also be used in calculating the statistical properties of the response to the noise portion of the input because, as will be shown, the statistical calculations may be considered to represent the solution of a transient problem.

Statistical techniques have received a wider application in the field of communications than in the field of feedback control. One reason for this is that in communications the problems encountered have more of a statistical nature. It is often desirable to treat in a statistical manner the signal portion of the input to a communication system as well as the noise portion, because the signal portion generally has such a complex nature that it is very difficult to work with a complete plot and it is usually of such high frequency with respect to the period of measurement that an autocorrelation of a sample can be quite meaningful. Besides, communication applications generally require that the

output follow the input with far less accuracy than do feedback-control applications, and consequently it is often sufficient for the designer to obtain only statistical information concerning the response of a communication system to the signal portion of the input.

Saturation Due to Noise

It is generally quite important that a noise component of the reference input not produce excessive saturation of any of the stages of the feedback-control system. Calculation of the degree of saturation by statistical techniques is extremely difficult in the general case because it requires a knowledge of the amplitude distribution of the variable rather than merely its rms value. On the other hand, Newton[2] shows that one can get a reasonable engineering solution to this problem quite readily if one can assume that the noise input has a Gaussian distribution.

If the noise has a Gaussian distribution of amplitudes and its average value is zero, the density of the noise amplitudes is given by the well-known expression

$$\psi(x_n) = \frac{1}{\sqrt{2\pi}\,\sigma_n} e^{-(1/2)(x_n/\sigma_n)^2} \tag{1}$$

where x_n represents a given amplitude of noise and σ_n the rms value of the noise. The probability P that the magnitude of noise does not exceed a given amplitude A is thus equal to the area under the density curve, between the limits $-A < x_n < +A$; i.e.,

$$P = \int_{-A}^{+A} dx_n\, \psi(x_n) \tag{2}$$

If A represents the saturation level of an element through which the noise must pass, then the probability P_s that the noise will saturate the element, i.e., the probability that the magnitude of noise will exceed A, is obtained by subtracting Eq. (2) from unity.

$$P_s = 1 - \int_{-A}^{+A} dx_n\, \psi(x_n) \tag{3}$$

To apply Eq. (3) it is convenient to nondimensionalize the noise amplitudes and saturation level in terms of the rms value of the noise. Define the nondimensional noise amplitude as u_n and the nondimensional saturation level as α:

$$u_n = x_n/\sigma_n \tag{4}$$

$$\alpha = A/\sigma_n \tag{5}$$

Then the probability that the noise will saturate an element having a saturation level equal to α times the rms value of the noise is

$$P_s = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\alpha}^{+\alpha} du_n \, e^{-u_n^2/2} \tag{6}$$

A plot of this probability is given in Fig. 1, as a function of the nondimensional saturation level α.

For Fig. 1 to apply to any variable in the system, it is necessary that that variable have a Gaussian distribution in amplitudes. Generally, one can assume this condition holds if he assumes that the input is Gaussian and that the percentage of time the system is in saturation is quite small because (a) if the

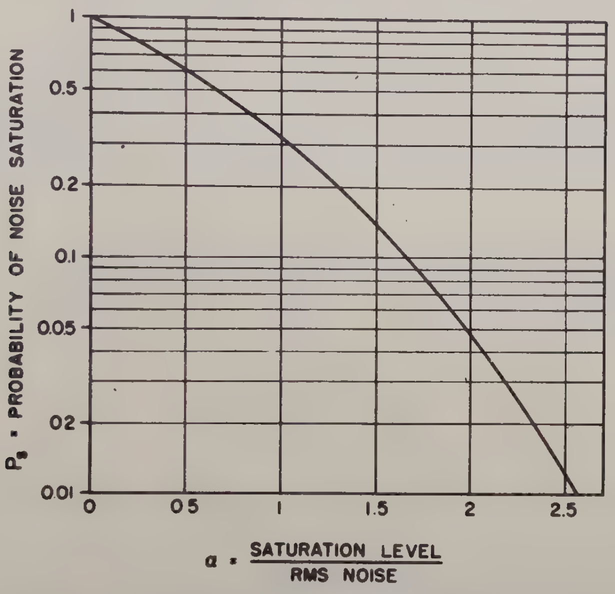


Fig. 1 - Plot for determining saturation due to noise.

per-unit time of saturation is small, it is logical to assume that the system behaves essentially as a linear system and (b) if the input to a linear system is Gaussian, the system response must be Gaussian.

Figure 1 can be employed as follows. Assume that the percentage time of saturation is to be limited to some value, say 5 per cent. Figure 1 shows that for a probability of saturation of 5 per cent the system must pass linearly noise amplitudes up to 1.96 times the rms value of the signal. Thus, to determine the noise level that must be passed linearly at any point in the system (in order to limit the probability of saturation by the noise, at any point, to the given value of 5 per cent), the rms value of the noise at that point should be computed on a linear basis and multiplied by the factor 1.96 to obtain the required linear range.

On the other hand, the signal as well as the noise must be considered in determining the required saturation level of the stages in a system. Adding the

noise amplitude which must be passed linearly to a time-varying plot of the signal component at a given stage gives a plot of the signal-plus-noise to be passed. In any particular case, engineering judgment can be used to determine whether the saturation level of that stage must be greater than the maximum value of this signal-plus-noise plot or whether a lower value can be allowed. There is a fundamental difference between the effects of saturation due to the noise portion and saturation due to the signal portion. Since the noise is of high frequency, it drives the system into saturation only for very short instants of time; whereas the signal portion, being of low frequency, could maintain saturation for such long periods of time that the system may not behave in a quasi-linear fashion even though the percentage of time during saturation is small.

BASIC EQUATIONS

The analysis to be presented is based upon the convolution integral described in detail in Appendix A. This integral gives the output time response $x_o(t)$ for a given input $x_i(t)$ as

$$x_o(t) = \int_0^\infty d\tau \; h(\tau) \; x_i(t - \tau) \tag{7}$$

where $h(\tau)$ is the impulse response of the system. The impulse response represents the inverse Laplace transform of the system transfer function $H(s)$, defined as

$$H(s) = \frac{X_o(s)}{X_i(s)} \tag{8}$$

where $X_o(s)$ and $X_i(s)$ are the transforms of $x_o(t)$ and $x_i(t)$ respectively.

There are four statistical functions pertaining to the variables $x_i(t)$ and $x_o(t)$, which are as follows: $\Phi_{ii}(\tau)$, the autocorrelation function of the input $x_i(t)$; $\Phi_{oo}(\tau)$, the autocorrelation function of the output $x_o(t)$; $\Phi_{io}(\tau)$ and $\Phi_{oi}(\tau)$, the cross-correlation functions of the input and the output. The equations for these functions are given below.

$$\phi_{ii}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} dt \; x_i(t) \; x_i(t + \tau) \tag{9}$$

$$\phi_{oo}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} dt \; x_o(t) \; x_o(t + \tau) \tag{10}$$

$$\phi_{io}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} dt \; x_i(t) \; x_o(t + \tau) \tag{11}$$

$$\phi_{oi}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} dt \; x_o(t) \; x_i(t + \tau) \tag{12}$$

Techniques for computing these functions from input and output data are described in Appendix A.

Some important characteristics of the correlation functions can be obtained by inspection. The equations for the autocorrelation functions $\Phi_{ii}(\tau)$ and $\Phi_{oo}(\tau)$ have the same value if τ is replaced by $-\tau$, so that

$$\phi_{ii}(\tau) = \phi_{ii}(-\tau) \tag{13}$$

$$\phi_{oo}(\tau) = \phi_{oo}(-\tau) \tag{14}$$

Thus, the autocorrelation functions are even functions of τ; the cross-correlation functions, on the other hand, are generally not. If in Eq. (11) the variable τ in $\Phi_{io}(\tau)$ is replaced by $-\tau$, the relation becomes equal to Eq. (12) for the other cross-correlation function $\Phi_{oi}(\tau)$. Thus,

$$\phi_{io}(\tau) = \phi_{oi}(-\tau) \tag{15}$$

If the autocorrelation function $\Phi_{ii}(\tau)$ in Eq. (9) is calculated for $\tau = 0$, it becomes

$$\phi_{ii}(0) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} dt \; x_i^2(t) \tag{16}$$

This is by definition the average of x_i^2 or, in other words, the mean-square value of the input, which is designated as $\overline{x_i^2}$. Thus,

$$\Phi_{ii}(0) = \overline{x_i^2} \tag{17}$$

and, likewise,

$$\phi_{oo}(0) = \overline{x_o^2} \tag{18}$$

EXPRESSION RELATING MEAN-SQUARE VALUE OF OUTPUT TO STATISTICAL CHARACTERISTICS OF INPUT

As has been shown, it is desirable to be able to calculate the mean-square value of the output response of a system from a statistical condensation of the input, rather than from the complete input plot. To find how this calculation

may be performed, consider the definition of the mean-square value of the output, which is

$$\overline{x_o^2} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} dt \; x_o(t) \; x_o(t) \tag{19}$$

Substitute into Eq. (19) for each of the variables $x_o(t)$ the convolution integral of Eq. (7), using separate time variables τ_1 and τ_2 to differentiate between the two $x_o(t)$ variables.

$$\overline{x_o^2} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} dt \int_0^{\infty} d\tau_1 \; h(\tau_1) \; x_i(t-\tau_1) \int_0^{\infty} d\tau_2 \; h(\tau_2) \; x_i(t-\tau_2). \tag{20}$$

Reverse the order of integration.

$$\overline{x_o^2} = \int_0^{\infty} d\tau_1 \; h(\tau_1) \int_0^{\infty} d\tau_2 \; h(\tau_2) \left[\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} dt \; x_i(t-\tau_1) \; x_i(t-\tau_2) \right] \tag{21}$$

The expression within the brackets may be put in the form

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} dt \; x_i(t - \tau_1) \; x_i\left[(t - \tau_1) + (\tau_1 - \tau_2)\right] \tag{22}$$

As shown by Eq. (9), this is the autocorrelation function of the input with $(\tau_1 - \tau_2)$ representing the time shift; i.e., it is $\phi_{ii}(\tau_1 - \tau_2)$. Thus, Eq. (21) may be expressed as

$$\overline{x_o^2} = \int_0^{\infty} d\tau_1 \; h(\tau_1) \int_0^{\infty} d\tau_2 \; h(\tau_2) \; \phi_{ii}(\tau_1 - \tau_2) \tag{23}$$

Equation (23) expresses the mean-square value of the output response of a system as a function of the autocorrelation function of the input and the system impulse response. Thus, the equation shows that the autocorrelation function of a signal contains all the information concerning that signal that is needed for determining the mean-square value of the response to that signal of any linear system. Although Eq. (23) might appear rather complex to the reader, it can be solved readily by transient and transform techniques. One can show from the equation that the mean-square value of the output can be found by treating the input autocorrelation functions as a transient input to the system. On the other hand, it is easier to demonstrate this by directly treating the input autocorrelation function as a transient input and calculating the response of the system to that transient input.

RELATING CORRELATION FUNCTIONS BY TRANSIENT APPROACH

The correlation functions are functions of the time shift τ but can be conveniently considered also as functions of real time t, and the output autocorrelation function $\Phi_{oo}(\tau)$ can be calculated by transient techniques from the input autocorrelation function $\Phi_{ii}(\tau)$. The mean-square value of the output $\overline{x_o^2}$ can then be found by determining the value of $\Phi_{oo}(\tau)$ at zero τ.

Consider $\Phi_{ii}(\tau_1)$ as a function of the real-time variable τ_1 and an input to a system with the impulse response h(t). The time-response of the system to this transient input is given by the convolution integral of Eq. (7) as

$$\int_0^{\infty} d\tau_2\, h(\tau_2)\, \phi_{ii}(\tau_1 - \tau_2) \tag{24}$$

Using the general form for the input autocorrelation function given in Eq. (9), substitute into this integral the expression for $\Phi_{ii}(\tau_1 - \tau_2)$.

$$\int_0^{\infty} d\tau_2\, h(\tau_2) \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} dt\, x_i(t)\, x_i\left[t + (\tau_1 - \tau_2)\right] \tag{25}$$

Reverse the order of integration.

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} dt\, x_i(t) \left[\int_0^{\infty} d\tau_2\, h(\tau_2)\, x_i\left[(t + \tau_1) - \tau_2\right]\right] \tag{26}$$

The expression within the large brackets is a convolution of the input $x_i(t+\tau_1)$ with the system impulse response $h(\tau_2)$ and, hence, represents the output $x_o(t + \tau_1)$, since by Eq. (7)

$$x_o(t + \tau_1) = \int_0^{\infty} d\tau_2\, h(\tau_2)\, x_i\left[(t + \tau_1) - \tau_2\right] \tag{27}$$

Substituting Eq. (27) into Eq. (26) gives

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} dt\, x_i(t)\, x_o(t + \tau_1) \tag{28}$$

Comparing this with Eq. (11) shows that it represents the cross-correlation function $\Phi_{io}(\tau_1)$. Thus, if the input autocorrelation function $\Phi_{ii}(\tau)$ is taken as a transient time-input to the system, the system time-response to this input is the cross-correlation function $\Phi_{io}(\tau)$.

By rotating the cross-correlation function $\Phi_{io}(\tau)$ about the zero time-shift axis, the cross-correlation function $\Phi_{oi}(\tau)$ is formed. Now determine the system response if $\Phi_{oi}(\tau)$ is taken as a transient input to the system. The convolution

integral of Eq. (7) gives as the system time-response to the transient input $\phi_{oi}(\tau_1)$,

$$\int_0^{\infty} d\tau_3 \; h(\tau_3) \; \phi_{oi}(\tau_1 - \tau_3) \tag{29}$$

Substitute into Eq. (29) the expression for $\phi_{oi}(\tau_1 - \tau_3)$ obtained from the general definition of Eq. (12).

$$\int_0^{\infty} d\tau_3 \; h(\tau_3) \quad \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} dt \; x_o(t) \; x_i\left[t + (\tau_1 - \tau_3)\right] \tag{30}$$

Reverse the order of integration.

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} dt \; x_o(t) \quad \left[\int_0^{\infty} d\tau_3 \; h(\tau_3) \; x_i\left[(t + \tau_1) - \tau_3\right]\right] \tag{31}$$

The expression within the large brackets is the convolution of the input $x_i(t + \tau_1)$ with the system impulse response $h(\tau_3)$ and hence represents the output $x_o(t + \tau_1)$. Thus, Eq. (31) becomes

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} dt \; x_o(t) \; x_o(t + \tau_1) \tag{32}$$

which is the output autocorrelation function $\phi_{oo}(\tau_1)$, given in Eq. (10). Thus, if the cross-correlation function $\phi_{oi}(\tau)$ is taken as a transient time-input to the system, the system-time response to this input is the output autocorrelation function $\phi_{oo}(\tau)$.

The complete transient method for calculating $\phi_{oo}(\tau)$ from $\phi_{ii}(\tau)$ is illustrated in Fig. 2. The input autocorrelation function $\phi_{ii}(\tau)$ is shown in Fig. 2(a). It is symmetric about zero τ, and its value at $\tau = 0$ is the mean-square value of the input $\overline{x_i^2}$. With $\phi_{ii}(\tau)$ as the transient input to the system H(s), the output is $\phi_{io}(\tau)$, which in general is not symmetric about zero τ.

Rotating the plot of $\phi_{io}(\tau)$ about the zero τ axis gives the plot of $\phi_{oi}(\tau)$ shown in Fig. 2(b). With $\phi_{oi}(\tau)$ as the transient input to the system H(s), the output is $\phi_{oo}(\tau)$, which is symmetric about the zero τ axis. The value of $\phi_{oo}(\tau)$ at zero τ is $\overline{x_o^2}$, the mean-square value of the output.

The transient calculations described above for obtaining $\phi_{oo}(\tau)$ from $\phi_{ii}(\tau)$ can be performed readily by the graphical method.[1] It is also possible to perform them by an analog computer, but this may be quite difficult because it requires that input-table plots of both $\phi_{ii}(\tau)$ and $\phi_{oi}(\tau)$ be prepared. Consequently, when an analog computer is to be employed, it is often best to modify the calculations as described in Appendix C so that only an input-table plot of $\phi_{ii}(\tau)$ need be prepared.

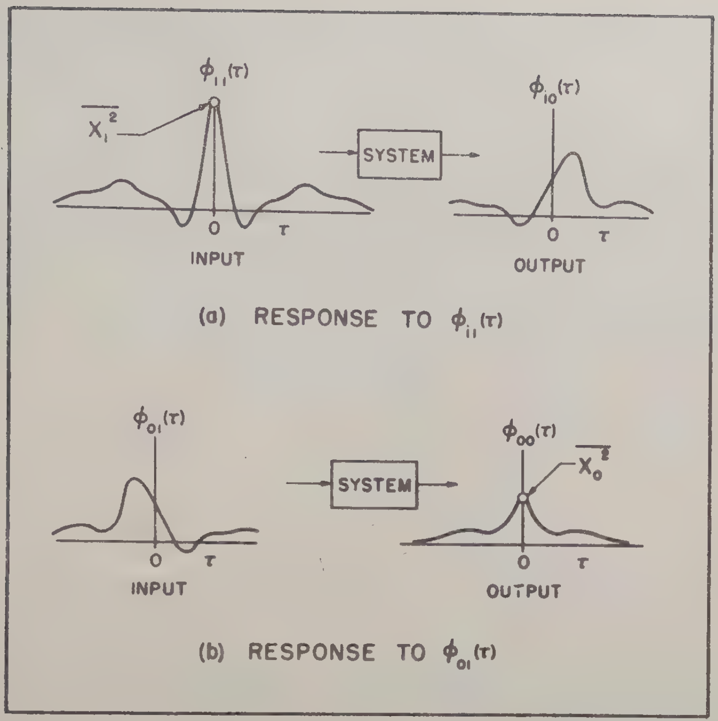


Fig. 2 - Transient response to correlation functions.

RELATING CORRELATION FUNCTIONS BY TRANSFORM METHOD

The mean-square value of the output may be calculated by transform techniques. In this section, the following basic equation is developed for the mean-square value of the output.

$$\overline{x_o^2} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega \left| H(j\omega) \right|^2 \Phi_{ii}(\omega) \tag{33}$$

where $H(j\omega)$ is the transfer function of the system and $\varphi_{ii}(\omega)$ is the Fourier transform of the input autocorrelation function $\phi_{ii}(\tau)$. Thus, in the transform procedure for calculating the mean-square value of the output, the transform of the input autocorrelation function is calculated and multiplied by the square of the magnitude plot of the system transfer function; the mean-square output is then proportional to the area under the resultant curve.

Empirically computed autocorrelation functions are necessarily of finite length, and the finite length is an important cause of error in the transform method. Techniques for performing transform computation in a way to minimize this error are given by Ross.[3] The finite length of the autocorrelation function also causes errors in the transient method, but it appears that the errors should be less severe when the transient method is used. However, a detailed study of the problem is required before this conclusion can be substantiated adequately.

The main advantage of the transform approach over the transient approach is that the transform of a correlation function expresses the data in terms of its frequency information, which often is more convenient to handle in an engineering analysis.

Since the correlation functions are not zero for negative values of τ, it is necessary to transform them by the Fourier transform rather than by the Laplace transform. The Fourier transform and its inverse are as follows.

$$F(\omega) = \int_{-\infty}^{+\infty} f(t)\, e^{-j\omega t}\, dt \tag{34}$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega)\, e^{-j\omega t}\, d\omega \tag{35}$$

The factor (2π) is sometimes placed with the $F(\omega)$ integral rather than with the $f(t)$ integral, but this makes no difference in the over-all calculations provided the pair of equations is consistent.

The Fourier transforms of the autocorrelation functions $\phi_{ii}(\tau)$ and $\phi_{oo}(\tau)$ are called the spectral densities or, sometimes, power spectra and are designated as $\Phi_{ii}(\omega)$ and $\Phi_{oo}(\omega)$. Thus,

$$\Phi_{ii}(\omega) = \int_{-\infty}^{\infty} \phi_{ii}(\tau)\, e^{-j\omega\tau}\, d\tau \tag{36}$$

$$\phi_{ii}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_{ii}(\omega)\, e^{-j\omega\tau}\, d\omega . \tag{37}$$

The Fourier transforms of the cross-correlation functions $\phi_{io}(\tau)$ and $\phi_{oi}(\tau)$ in like manner are designated at $\Phi_{io}(\omega)$ and $\Phi_{oi}(\omega)$, respectively.

It is possible to calculate the spectral density by transforming the time function directly, rather than by transforming its autocorrelation function. Designate the Fourier transform of the time function $x(t)$ measured over the time interval $-T < t < +T$ as $X_T(\omega)$. The spectral density of $x(t)$, designated $\Phi(\omega)$, can than be shown to be equal to

$$\Phi(\omega) = \lim_{T\to\infty} \frac{1}{2T} X_T(\omega)\, X_T^*(\omega) \tag{38}$$

$$= \lim_{T\to\infty} \frac{1}{2T} \left| X_T(\omega) \right|^2 . \tag{39}$$

Equation (39) is proved in Appendix B. Note that the asterisk in Eq. (38) designates the conjugate of X_T.

Theoretically, the spectral densities could be measured directly from the input and output signals by means of a narrow bandwidth, variable-frequency filter. If the input signal x_i is passed through a narrow bandwidth filter centered at the frequency ω_1, the average value of the output from the filter, divided by the bandwidth of the filter, approaches $\Phi_{ii}(\omega_1)$ as the bandwidth approaches zero. On the other hand, this method of measurement is often quite difficult to perform in practice and is subject to error because of nonideal characteristics of the filter. However, the concept is important primarily because of the physical significance it gives to the spectral density.

Since the time response of the system to the transient input $\phi_{ii}(\tau)$ is $\phi_{io}(\tau)$, the transforms of these correlation functions must be related by

$$\Phi_{io}(\omega) = H(j\omega)\ \Phi_{ii}(\omega) \tag{40}$$

where $H(j\omega)$ is the system frequency response. Similarly, since the time-response of the system to the transient input $\phi_{oi}(\tau)$ is $\phi_{oo}(\tau)$, then

$$\Phi_{oo}(\omega) = H(j\omega)\ \Phi_{oi}(\omega). \tag{41}$$

Now, it can be shown readily from the basic Fourier transform relations in Eqs. (34) and (35) that

$$F(-\omega) = F^*(\omega) \tag{42}$$

and if $f_2(t)$ is equal to $f_1(-t)$, then the transforms are related by

$$F_2(\omega) = F_1(-\omega). \tag{43}$$

Therefore, the cross-correlation transforms are related to each other by

$$\Phi_{oi}(\omega) = \Phi_{io}(-\omega) = \Phi^*_{io}(\omega) \tag{44}$$

since $\phi_{oi}(\tau)$ is equal to $\phi_{io}(-\tau)$. Substitute Eq. (44) into Eq. (41).

$$\Phi_{oo}(\omega) = H(j\omega)\ \Phi^*_{io}(\omega) \tag{45}$$

Substitute for $\Phi^*_{io}(\omega)$ the conjugate of Eq. 40.

$$\Phi_{oo}(\omega) = H(j\omega)\ H^*(j\omega)\ \Phi^*_{ii}(\omega) \tag{46}$$

Now, since $\phi_{ii}(\tau)$ is equal to $\phi_{ii}(-\tau)$, then by Eqs. (42) and (43)

$$\Phi^*_{ii}(\omega) = \Phi_{ii}(-\omega) = \Phi_{ii}(\omega)\ . \tag{47}$$

Substitute Eq. (47) into Eq. (46).

$$\Phi_{oo}(\omega) = H(j\omega)\ H^*(j\omega)\ \Phi_{ii}(\omega) \tag{48}$$

$$= \left\| H(j\omega) \right\|^2 \Phi_{ii}(\omega) \tag{49}$$

Thus, the output spectral density is equal to the product of the square of the magnitude of the system transfer function times the input spectral density.

There are a number of convenient relations between the spectral density and the autocorrelation function that can be obtained by inspection from Eqs. (36) and (37). Setting ω equal to zero in Eq. (36) gives

$$\Phi_{ii}(0) = \int_{\infty}^{\infty} \phi_{ii}(\tau)\, d\tau . \tag{50}$$

Thus, the dc value (zero-frequency value) of the spectral density is equal to the area under the autocorrelation function. Setting τ equal to zero in Eq. (37) gives

$$\overline{x_i^{\,2}} = \phi_{ii}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_{ii}(\omega)\, d\omega \tag{51}$$

$$= \int_{-\infty}^{\infty} \Phi_{ii}(f)\, df \tag{52}$$

where f is the frequency in cycles per second. Thus, the area under the spectral-density curve (with f as the frequency variable) is equal to the value of the autocorrelation function at zero τ, which is the mean-square value of the signal. To calculate the mean-square value of the output, apply Eq. (51) to the output.

$$\overline{x_o^{\,2}} = \phi_{oo}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_{oo}(\omega)\, d\omega . \tag{53}$$

Substitute for $\Phi_{oo}(\omega)$ the expression in Eq. (48).

$$\overline{x_o^{\,2}} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left| H(j\omega) \right|^2 \Phi_{ii}(\omega)\, d\omega \tag{33}$$

$$= \int_{-\infty}^{\infty} \left| H(j2\pi f) \right|^2 \Phi_{ii}(f)\, df \tag{54}$$

Equation (33) was the basic equation given above for calculating the mean-square output.

In computing the transforms, it is much easier to express the equations as integrations of real variables, rather than as integrations of complex variables. This can be performed quite readily with the autocorrelation functions, since they are even functions. The transform integral of Eq. (36) can be expressed as

$$\Phi_{ii}(\omega) = \int_{-\infty}^{\infty} \phi_{ii}(\tau)\,(\cos \omega\tau - j \sin \omega\tau)\, d\tau \tag{55}$$

because

$$e^{-j\theta} = \cos\theta - j \sin\theta . \tag{56}$$

Since $\Phi_{ii}(\tau)$ is an even function of τ, the sine portion of Eq. (55) is zero, and the equation reduces to

$$\Phi_{ii}(\omega) = \int_{-\infty}^{\infty} \Phi_{ii}(\tau) \cos \omega\tau \, d\tau \tag{57}$$

$$= 2 \int_{0}^{\infty} \Phi_{ii}(\tau) \cos \omega\tau \, d\tau \, . \tag{58}$$

This integral is always real, and so the phase of the spectral density must always be zero. Besides $\Phi_{ii}(\omega)$ must be symmetric about the zero-frequency axis. Thus,

$$\Phi_{ii}(\omega) = \Phi_{ii}(-\omega) = \Phi^{*}_{ii}(\omega) \, . \tag{59}$$

Since the spectral density is an even function of frequency, it can be shown readily that the inverse transform integral of Eq. (37) reduces to

$$\Phi_{ii}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_{ii}(\omega) \cos \omega\tau \, d\omega \tag{60}$$

$$= \frac{1}{\pi} \int_{0}^{\infty} \Phi_{ii}(\omega) \cos \omega\tau \, d\omega \, . \tag{61}$$

APPENDIX A

Graphical Interpretation of Convolution and Correlation Integrals

Convolution Integral

The convolution integral can be derived by physical reasoning. Consider the input $x_i(t)$ as representing the sum of a large number of pulses of width Δt and magnitude $x_i(t)$. If the time interval Δt is sufficiently small with respect to the response time of the system, each pulse has the effect of an impulse of the same area, $[x_i(t)\,\Delta t]$. Figure 3(a) shows the response of a system to a unit impulse. Figure 3(b) shows an input $x_i(t)$ to the system and one of the pulses into which $x_i(t)$ is considered to be divided. The response to this pulse is obtained by multiplying the unit impulse response $h(\tau)$ by $[x_i(t)\,\Delta t]$ to obtain the response curve shown in Fig. 3(b). At the time t_1 the response to this pulse has the value

$$\left[x_i(t)\,\Delta t\right] \, h(t_1 - t) \tag{62}$$

as shown in the figure. The total response $x_o(t_1)$ to the input $x_i(t)$ at the time t_1 is obtained by adding together the responses to all the pulses into which $x_i(t)$ is divided

$$x_o(t_1) = \sum \left[x_i(t)\,\Delta t\right] \, h(t_1 - t) \tag{63}$$

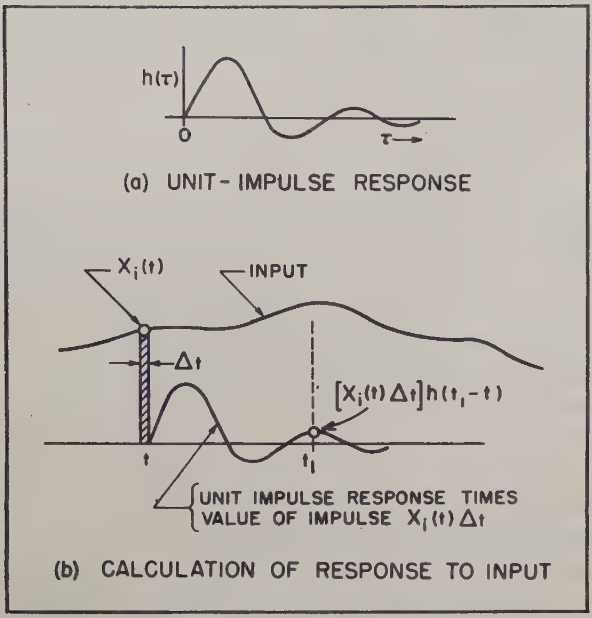


Fig. 3 - Derivation of convolution integral.

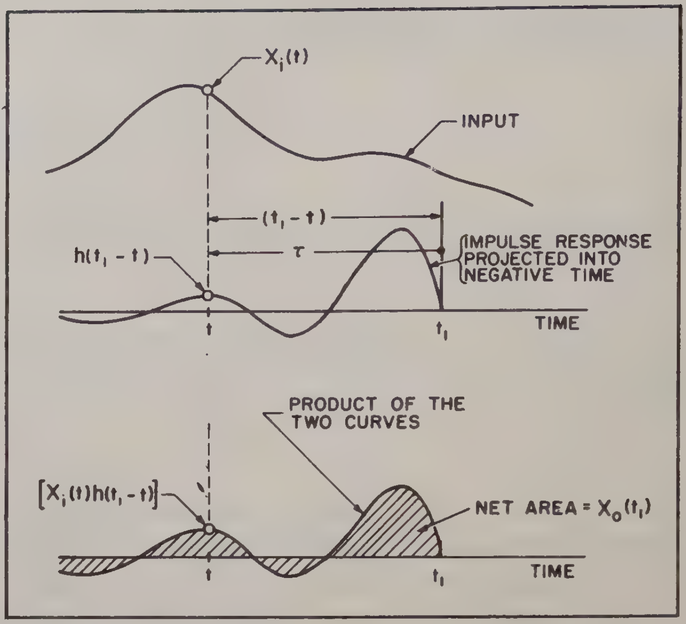


Fig. 4 - Alternate interpolation of convolution process.

for all the pulses between $t = -\infty$ and $t = t_1$. Passing to the limit gives

$$x_o(t_1) = \int_{-\infty}^{t_1} dt\, x_i(t)\, h(t_1 - t). \tag{64}$$

The convolution process is illustrated in Fig. 4 in a slightly different manner than in Fig. 3. The time t_1 at which the output is being calculated can be considered to represent the time of observation. The unit impulse response $h(\tau)$ is shown projected from the time of observation t_1 into the direction of past (or negative) time and hence represents $h(t_1 - t)$ as shown. Multiplying the input curve $x_i(t)$ point by point by the impulse response $h(t_1-t)$ gives the product curve $x_i(t)\, h(t_1-t)$. By Eq. (64) the response x_o at present time t_1 is the integral of this product curve from $-\infty$ up to the present time t_1 and hence represents the net area under the product curve.

The construction of Fig. 4 shows why the unit impulse response $h(\tau)$ is termed a "weighting function." The impulse response $h(\tau)$ multiplies or "weights" the past values of the input. For systems with a finite bandwidth the impulse response $h(\tau)$ goes to zero at infinite τ, and hence the values of the input $x_i(t)$ far in the past must have little effect upon the present output $x_o(t_1)$. The more recent values of the input are, in general, weighted much more than long-past values. It is obvious also that values of the input which have not yet occurred cannot affect the present value of the output. This condition is assured in the convolution integral because the impulse response is zero for negative τ.

The variable τ described above is equal to

$$\tau = (t_1 - t) \tag{65}$$

and therefore can be considered to represent elapsed time, a time variable from present time t_1 projected into the past as shown in Fig. 4. To express the convolution integral in terms of τ, note that by Eq. (65)

$$d\tau = -\, dt. \tag{66}$$

Substituting Eqs. (65) and (66) into Eq. (64) gives

$$x_o(t_1) = \int_{(t_1-\tau) = -\infty}^{(t_1-\tau) = t_1} (-d\tau)\; x_i(t_1-\tau)\; h(\tau) \; . \tag{67}$$

Expressing the limits in terms of τ gives

$$x_o(t_1) = \int_0^{\infty} d\tau \; h(\tau)\; x_i(t_1 - \tau) \; . \tag{68}$$

Equation (68) is the more convenient form of the convolution integral which is used in this paper.

Autocorrelation Function

The process of computing an autocorrelation function is illustrated in Fig. 5. Figure 5(a) shows a plot of an input function $x_i(t)$ and shows the same plot shifted to the right by the time shift τ_1 to obtain $x_i(t + \tau_1)$. These two curves, $x_i(t)$ and $x_i(t + \tau_1)$, are multiplied together point by point to obtain the product curve $[x_i(t)\; x_i(t + \tau_1)]$, shown in Fig. 5(b). The average value of this product curve represents the autocorrelation function $\phi_{ii}(\tau_1)$. Thus, $\phi_{ii}(\tau_1)$ is

$$\phi_{ii}(\tau_1) = \text{Average} \left\{ x_i(t)\; x_i(t + \tau_1) \right\} \tag{69}$$

$$= \lim_{(t_2-t_1) \to \infty} \left\{ \frac{1}{(t_2-t_1)} \int_{t_1}^{t_2} x_i(t)\; x_i(t+\tau_1)\; dt \right\} \tag{70}$$

In practice, of course, the averaging is always performed over a finite period of time $(t_2 - t_1)$. However, if this period of time $(t_2 - t_1)$ is much greater than the maximum time shift τ_1, the point $\phi_{ii}(\tau_1)$ of the autocorrelation curve may be considered to be approximated adequately.

The expression for $\phi_{ii}(\tau_1)$ given in Eq. (70) is equivalent to the more common form

$$\phi_{ii}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} dt \; x_i(t)\; x_i(t + \tau) \tag{71}$$

The averaging process of the product curve described in Fig. 5(b) gives one point $\phi_{ii}(\tau_1)$ on the convolution curve, as shown in Fig. 5(c). It is obvious that if the interval $(t_2 - t_1)$ may be considered infinite, it makes no difference

in Fig. 5(a) if the second curve is shifted in the positive or negative direction by the amount τ_1. In either case there are two identical curves shifted from one another by the amount τ_1. Thus, the autocorrelation function $\Phi_{ii}(\tau)$ must have the same value at $-\tau$ that it has at $+\tau$; i.e., $\Phi_{ii}(\tau)$ must be symmetric about the zero τ axis.

If the input $x_i(t)$ does not have any periodic components, the variations in $x_i(t)$ are uncorrelated for infinite τ shifts. Consequently, as τ approaches

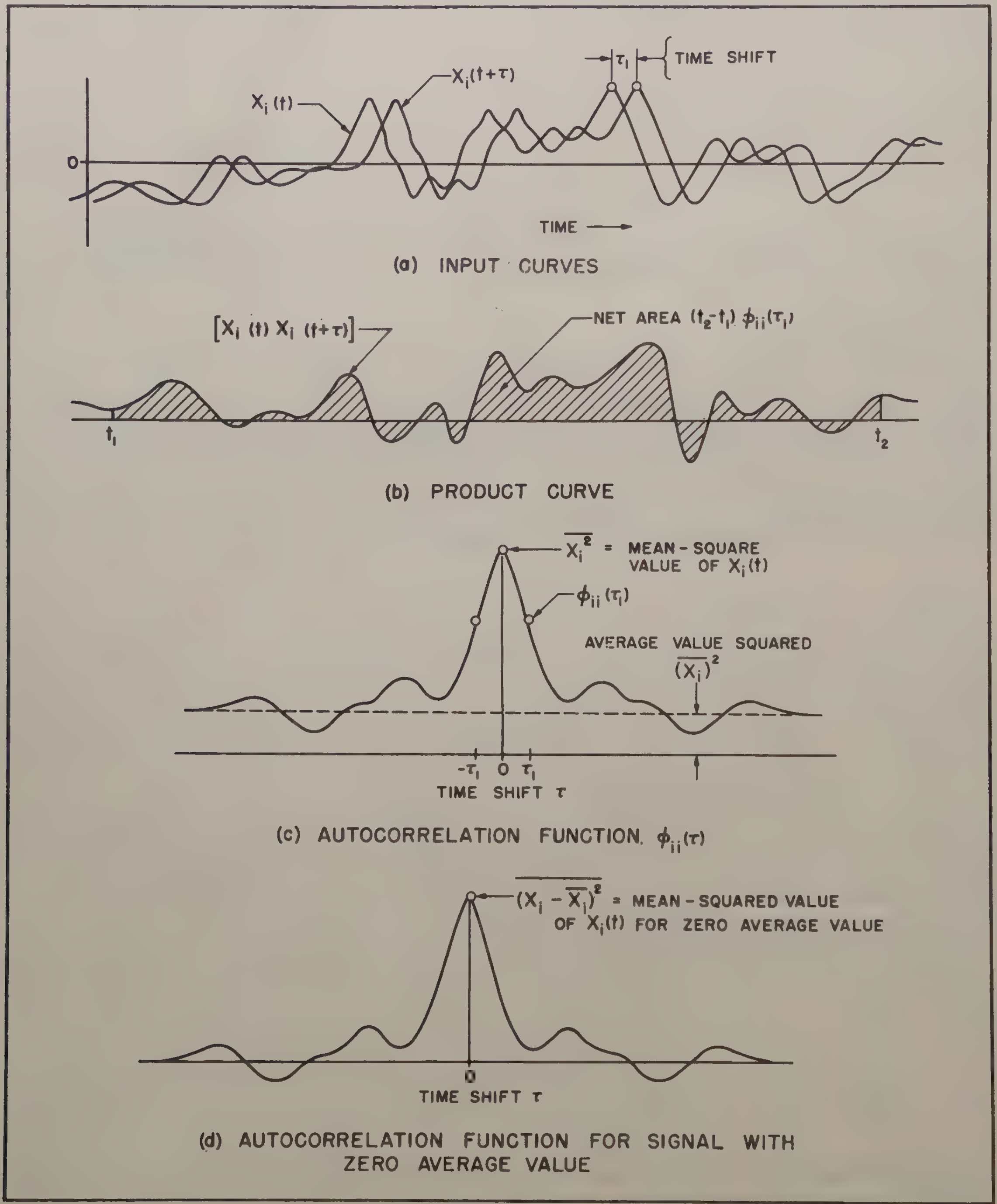


Fig. 5 - Description of autocorrelation procedure.

infinity the autocorrelation function $\phi_{ii}(\tau)$ approaches a constant value, and that constant value merely represents the square of the average value of the input x_i. Symbolically, this final value of $\phi_{ii}(\tau)$ is $(\overline{x_i})^2$.

In practice the final value of $\phi_{ii}(\tau)$ is almost always of no interest. It is therefore removed, either by subtracting from the original input curve $x_i(t)$ its average value $\overline{x_i}$ or by subtracting from the autocorrelation function $\phi_{ii}(\tau_1)$ the square of the average value of the input $(\overline{x_i})^2$. In either case the autocorrelation function actually used has the form of Fig. 5(d), with a zero final value.

This practice of subtracting out the average value of the input is justified as follows. In most cases the average value of the signal considered really is zero, and it appears in the computation only because a bias is developed by the original measurement equipment or in the device computing the autocorrelation function. Besides, if there really is a steady component (average x_i) in the input signal, the response of the system to the steady component could be calculated best by considering the steady component separately from the rest of the input.

It is apparent that this computation of the autocorrelation function could be extremely complicated if performed manually. Consequently, the calculation of autocorrelation functions is usually performed by a computer, except for some very specialized types of input functions in which analytical calculation is possible. Very often only the positive half of the autocorrelation function (i.e., for positive τ) is given. This is because the negative half is merely its mirror image and, hence, is not necessary. Nevertheless, one should always consider the autocorrelation function as having a symmetric negative portion. In this sense, it is quite unlike a system impulse response $h(\tau)$, which must be zero for negative τ. That is why the Fourier transform is applied to $\phi_{ii}(\tau)$, whereas the Laplace transform can be used with $h(\tau)$.

Cross-Correlation Function

The cross-correlation function $\phi_{io}(\tau)$ is obtained in the same manner as the autocorrelation function $\phi_{ii}(\tau)$, except that the input $x_i(\tau)$ is multiplied by the shifted output $x_o(t + \tau)$ and the average of the resultant product curve calculated. Thus,

$$\phi_{io}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} dt\, x_i(t)\, x_o(t + \tau). \tag{72}$$

If the output curve $x_o(t)$ is fixed and $x_i(t)$ shifted forward by τ to obtain $x_i(t + \tau)$, the other cross-correlation function $\phi_{oi}(\tau)$ is obtained, which is

$$\phi_{oi}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} dt\, x_o(t)\, x_i(t + \tau). \tag{73}$$

It is obvious that the same value is obtained by shifting the output curve forward in time by τ_1 or by shifting the input curve backward in time by τ_1. Hence,

$$\phi_{io}(\tau) = \phi_{oi}(-\tau). \tag{74}$$

On the other hand, the sense of the relative shift between the input and output curves (i.e., which one is shifted forward) does make a difference. Therefore, the cross-correlation functions are generally not symmetric about the zero axis; i.e., $\Phi_{io}(\tau)$ is generally not equal to $\Phi_{io}(-\tau)$.

In calculating the cross-correlation functions, the average values of both the input $x_i(t)$ and output $x_o(t)$ curves are generally subtracted out, so that the final values of the cross-correlation functions approach zero at infinite τ (provided there is no common periodicity in the two signals). If these average values were not subtracted out, the cross-correlation functions would approach at infinite τ the value $(\overline{x_i})(\overline{x_o})$, which is the product of the average values of the two signals $x_i(t)$ and $x_o(t)$. If the cross-correlation function approaches a periodic waveform at large τ, this indicates a common periodicity in the two signals.

APPENDIX B

Relation Between Spectral Density and Transform of Time Function

The following is a demonstration of the relation given in Eq. (39), which relates the spectral density to the transform of the actual time-varying signal rather than to the autocorrelation of that time signal.

Define the function $x_T(t)$ as equal to the time signal $x(t)$ over the time region $-T < t < +T$ and equal to zero outside that region. The transform of $x_T(t)$ is $X_T(\omega)$.

$$X_T(\omega) = \int_{-\infty}^{\infty} dt \; x_T(t) \, e^{-j\omega t} \tag{75}$$

A correlation function $\Phi_T(\tau)$ may be defined in terms of $x_T(t)$ as follows.

$$\phi_T(\tau) = \frac{1}{2T} \int_{-\infty}^{\infty} x_T(t) \, x_T(t + \tau) \, dt \tag{76}$$

The autocorrelation function $\Phi(\tau)$ of the total signal $x(t)$ is the limit of $\Phi_T(\tau)$ as the time region becomes infinite.

$$\phi(\tau) = \lim_{T \to \infty} \phi_T(\tau) \tag{77}$$

The transform of $\Phi_T(\tau)$ is designated as $\Phi_T(\omega)$ and is equal to

$$\Phi_T(\omega) = \int_{-\infty}^{\infty} d\tau_1 \; \Phi_T(\tau_1) \, e^{-j\omega\tau_1} \; . \tag{78}$$

Therefore, the spectral density of the complete input $x(t)$, which is designated $\Phi(\omega)$, is the limit of $\Phi_T(\omega)$ as the time region becomes infinite.

$$\Phi(\omega) = \lim_{T \to \infty} \Phi_T(\omega) \tag{79}$$

Substitute into Eq. (78) the expression in Eq. (76) for $\Phi_T(\tau_1)$.

$$\Phi_T(\omega) = \int_{-\infty}^{\infty} d\tau_1 \, e^{-j\omega\tau_1} \frac{1}{2T} \int_{-\infty}^{\infty} dt \, x_T(t) \, x_T(t + \tau_1) \tag{80}$$

Now, $e^{-j\omega\tau_1}$ can be expressed as

$$e^{-j\omega\tau_1} = e^{-j\omega(t + \tau_1)} \, e^{j\omega t} \tag{81}$$

so that Eq. (80) is equal to

$$\Phi_T(\omega) = \frac{1}{2T} \int_{-\infty}^{\infty} d\tau_1 \int_{-\infty}^{\infty} dt \, e^{j\omega t} \, x_T(t) \, e^{-j\omega(t + \tau_1)} x_T(t+\tau_1). \tag{82}$$

Reverse the order of integration.

$$\Phi_T(\omega) = \frac{1}{2T} \int_{-\infty}^{\infty} dt \, e^{j\omega t} \, x_T(t) \int_{-\infty}^{\infty} d\tau_1 \, e^{-j\omega(t+\tau_1)} x_T(t+\tau_1) \tag{83}$$

In terms of the second integral the variable t is a constant; therefore, for that integral $d\tau_1$ is equal to

$$d\tau_1 = d(t + \tau_1) \, . \tag{84}$$

Substitute for $(t + \tau_1)$ in the second integral of Eq. (83) the variable τ_2. Equation (83) becomes

$$\Phi_T(\omega) = \frac{1}{2T} \int_{-\infty}^{\infty} dt \, e^{j\omega t} \, x_T(t) \int_{-\infty}^{\infty} d\tau_2 \, e^{-j\omega\tau_2} \, x_T(\tau_2) \, . \tag{85}$$

Comparing the two integrals with the expression in Eq. (75) for $X_T(\omega)$ shows that the second integral is the transform of $x_T(t)$ which is $X_T(\omega)$, and the first is the conjugate of this transform which is $X_T^*(\omega)$. Thus,

$$\Phi_T(\omega) = \frac{1}{2T} X_T^*(\omega) \, X_T(\omega) \, . \tag{86}$$

Since the limit of $\Phi_T(\omega)$ is the spectral density $\Phi(\omega)$, then $\Phi(\omega)$ is

$$\Phi(\omega) = \lim_{T \to \infty} \frac{1}{2T} X_T^*(\omega) \, X_T(\omega) \tag{87}$$

$$= \lim_{T \to \infty} \frac{1}{2T} \left| X_T(\omega) \right|^2 \tag{88}$$

which was to be shown.

Method for Computing Output Autocorrelation Function on an Analog Computer

In calculating the mean-square value of the output signal for a given input autocorrelation function, analog computers can be very useful, especially if a great many settings of the system parameters are to be tried. This calculation could be performed by an analog computer with a two-step procedure: first, determine the response $\phi_{io}(\tau)$ of a simulated system to the input $\phi_{ii}(\tau)$; and second, reverse the plot of $\phi_{io}(\tau)$ in time to get $\phi_{oi}(\tau)$ and find its response, which is $\phi_{oo}(\tau)$. However, this procedure may be quite inconvenient because it requires that an input-table plot of $\phi_{oi}(\tau)$ be prepared for each parameter setting of the system. On the other hand, the procedure can be modified as described below, so that the output autocorrelation function can be calculated with only a plot of $\phi_{ii}(\tau)$ required for the input table.

Since the output autocorrelation function $\phi_{oo}(\tau)$ is the time response of the system with the cross-correlation function $\phi_{oi}(\tau)$ as a transient input, then $\phi_{oo}(\tau)$ can be expressed in terms of $\phi_{oi}(\tau)$ by the convolution integral.

$$\phi_{oo}(\tau_1) = \int_0^\infty d\tau_2\ h(\tau_2)\ \phi_{oi}(\tau_1 - \tau_2) \tag{89}$$

Since $\phi_{oi}(\tau)$ is equal to $\phi_{io}(-\tau)$, then

$$\phi_{oi}(\tau_1 - \tau_2) = \phi_{io}(\tau_2 - \tau_1). \tag{90}$$

Substituting Eq. (90) into Eq. (89) gives

$$\phi_{oo}(\tau_1) = \int_0^\infty d\tau_2\ h(\tau_2)\ \phi_{io}(\tau_2 - \tau_1). \tag{91}$$

For τ_1 equal to zero the output <u>aut</u>ocorrelation function $\phi_{oo}(\tau_1)$ represents the mean-square value of the output $\overline{x_o^2}$, and Eq. (91) becomes

$$\overline{x_o^2} = \phi_{oo}(0) = \int_0^\infty d\tau_2\ h(\tau_2)\ \phi_{io}(\tau_2). \tag{92}$$

Figure 6 illustrates graphically how Eq. (92) may be computed. The response of the system to the transient input $\phi_{ii}(\tau)$ is $\phi_{io}(\tau)$, shown in Fig. 6(b). The response of the system to a unit impulse at $\tau = 0$ is the unit impulse response $h(\tau)$, shown in Fig. 6(c). Multiplying point by point the plots of $\phi_{io}(\tau)$ and $h(\tau)$ gives the curve in Fig. 6(d). The area under this curve is the mean-square output $\overline{x_o^2}$.

Figure 7 shows the computer set-up for computing the mean-square output value. On the input table there is a plot of the input autocorrelation function $\phi_{ii}(\tau)$ and a trigger mark placed at the point of zero τ. The input autocorrelation function $\phi_{ii}(\tau)$ is fed to an analog of the system being studied, and the output is $\phi_{io}(\tau)$. The trigger actuates a pulse generator feeding a sharp pulse

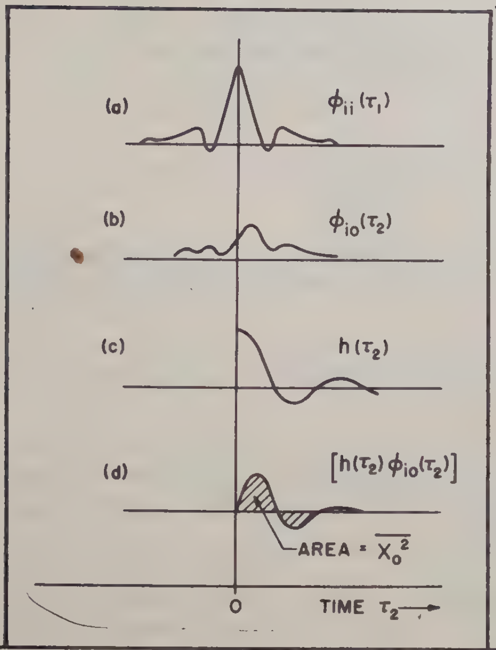
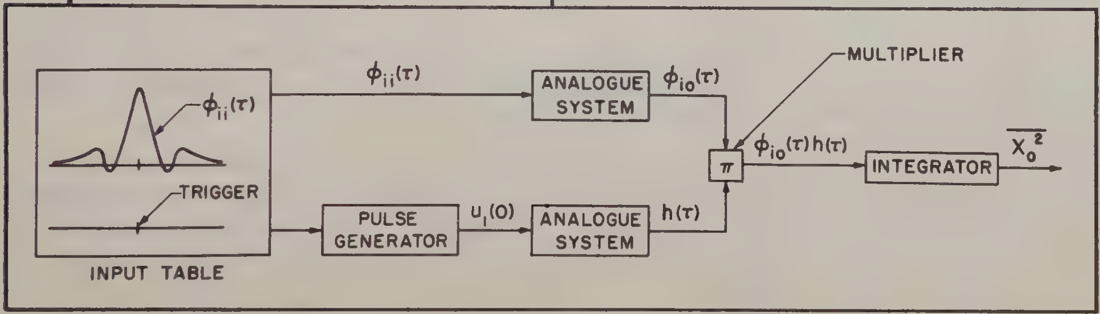


Fig. 6 (left) - Steps in computing mean-square output by analog computer.

Fig. 7 (below) - Analog computer set-up for computing mean-square value.



which is effectively a unit impulse into a second analog of the system, and the output is the impulse response $h(\tau)$. The outputs from the two analog systems are fed to a multiplier unit to form the product $\phi_{io}(\tau)\ h(\tau)$ and this is fed into an integrator, which delivers in the steady state the mean-square output $\overline{x_o^2}$.

One practical difficulty that the technique has is that it is difficult to use impulses in an analog computer without producing computer saturation. A simple way to alleviate the problem is to simulate in the impulse-response path the transfer function of the system multiplied by the factor s and feed into this, in place of the impulse, a unit step actuated by the trigger.

The mean-square value of the output, calculated from the computer set-up, represents the value of the output autocorrelation function for zero τ. To obtain other points on the output autocorrelation function plot, shift the trigger with respect to the input autocorrelation function by the desired time shift τ. That this is so can be seen by expressing Eq. (91) in the form

$$\phi_{oo}(\tau_1) = \int_0^\infty d\tau_2\ h(\tau_2 + \tau_1)\ \phi_{io}(\tau_2)\ . \qquad (93)$$

Thus, a shift of the trigger in the positive direction with respect to $\phi_{ii}(\tau)$ by the amount τ_1 produces the point on the output autocorrelation function at the positive time shift τ_1.

BIBLIOGRAPHY

1. Biernson, G. A.: "A simple method for calculating the time response of a system to an arbitrary input." Trans. AIEE, Paper No. 55-553 (1955).

2. Newton, G. C., Jr.: "Compensation of feedback-control systems subject to saturation." Journal of the Franklin Institute, Vol. 254, Nos. 4 and 5 (1952).

3. Ross, D. T.: "Improved computational techniques for Fourier transformation." Massachusetts Institute of Technology, Servomechanisms Laboratory, Report No. 7138-R-5 (June 1954).

4. Truxal, J. G.: "Control-System Synthesis." New York, McGraw-Hill Book Co., 1955, Chapters 7-8.

5. James, Nichols, and Phillips: "Theory of Servomechanisms." New York, McGraw-Hill Book Co., 1947, Massachusetts Institute of Technology Radiation Laboratory Series, Vol. XXV, Chapters 6-8.

A SURVEY OF TECHNIQUES FOR THE ANALYSIS OF SAMPLED-DATA CONTROL SYSTEMS

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Summary

The present use in control systems of pulsed-data links, track-while-scan radar, digital computers, and many other intermittently operative devices has stimulated interest in the analysis of sampled-data control systems. The tremendous effort now being expended to develop the techniques for analyzing such systems has resulted in the proposal of several methods of analysis. This paper presents an up-to-date discussion of the state of the art and lists proposed techniques of analysis with an explanation of the applications and limitations of each.

Introduction

The discussion which follows is concerned chiefly with the three principal methods of analysis currently exploited in the literature: impulse-response analysis, frequency-response analysis, and transfer-function analysis with the aid of the z-transformation.

In the very simple sampled-data feedback control system in Figure 1, the error E between desired output R and actual output C is sampled at discrete moments of time. The sampled error signal is represented by E*. The sampling may be either periodic or aperiodic; in the latter case, it may be dependent upon some characteristic of a signal, usually the system error. The impulse-response analysis is the only one of the three principal methods which is directly applicable to the aperiodically sampling system.

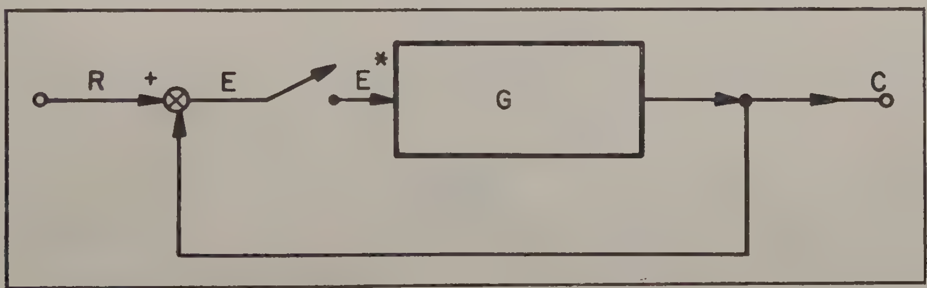


Fig. 1 - A simple sampled-data system.

Impulse Response

The impulse-response analysis is the most informative but usually the most laborious to apply of the three principal methods.[17] Between sampling

instants the system in Figure 1 operates as an open-loop control system with pulse inputs uniformly spaced in time. The instantaneous output can be approximated by a superposition of appropriately weighted impulse responses of the open-loop system, each impulse response commencing at the sampling instant for which its weighting factor was determined by the error sample.

The application of this procedure to a simple system with a unit-step input is illustrated in Figure 2. The sampling in this system is periodic with period T. The impulse at t = 0 is of unit amplitude, the magnitude of the impulse at the second sampling instant is equal to unity minus the magnitude of the response of the open-loop system to a unit impulse T seconds after the application of this impulse, etc.

Frequency Response

Impulse-response analysis becomes very laborious when the open-loop transfer function is of high order. For such systems, if the sampling is periodic, an extension of the conventional frequency-response-analysis

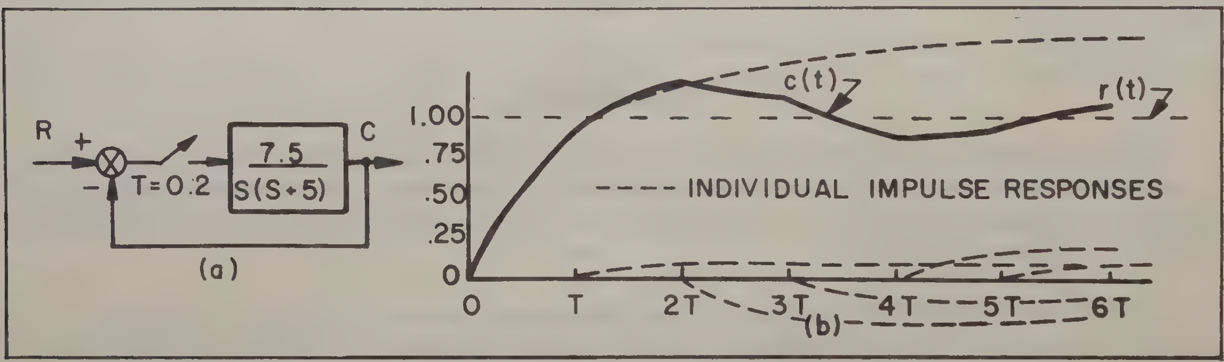


Fig. 2 - Response of a particular sampled-data system to a unit step input.

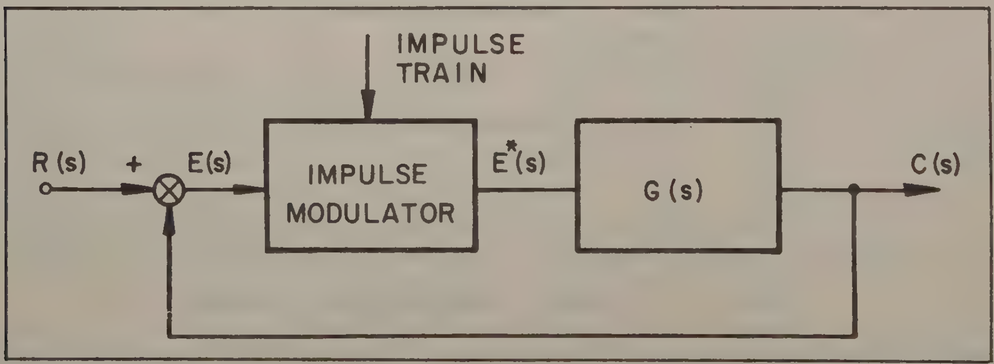


Fig. 3 - Representation of the sampling device as an inpulse modulator.

techniques may yield the desired information with less effort than the impulse-response analysis. The analysis in terms of frequency response, based upon the concept that the sampling device is an impulse modulator, is illustrated in Figure 3, which is a redrawing of Figure 1.

When considered in this light, the sampling device is seen to generate signals complementary to the complex-frequency components of its input signal. For an input at a single frequency, the output of the sampling device is found to possess a frequency spectrum containing the input frequency and an infinite number of so-called complementary frequencies separated from one another and from the input frequency by integral multiples of the sampling frequency. In general, a given component of the modulator input and all modulator output signals complementary to this component have the same amplitude at the modulator output. The expression which relates the sampled error to the continuous error is

$$E^* = \frac{1}{T} \sum_{n=-\infty}^{+\infty} E(s + jn\Omega) \quad \text{where } \Omega = \frac{2\pi}{T} \quad \text{is the sampling frequency.}$$

In a closed-loop system the complementary signals in the modulator output signal are transmitted to the input to the modulator. Because of the filtering action of the system, the components of the frequency spectrum of the signal at the input to the sampling device are not, in general, all of the same amplitude. Thus, it is possible to sustain a system output of the desired nature (i.e., one with a nonperiodic Fourier transform) even though the Fourier transform of the output of the sampling device must be periodic.

A complete frequency-response analysis of such a system must, of course, take into account the infinite number of complementary signals; as a result, an infinite series is involved. For practical purposes, however, use of a finite number of terms to approximate the infinite series yields satisfactory results. For most frequency-response analyses in which extreme accuracy is not required, all that need be considered are the system input signal and the complementary signal at a frequency equal to the frequency of the input signal minus the sampling frequency, since the magnitude of the transfer function at the other complementary frequencies is relatively small. If the sampling frequency is high and the filtering is fairly effective, satisfactory results can be obtained by ignoring all the complementary signals. [11]

The system used to illustrate the technique of impulse-response analysis is used again to illustrate the technique of frequency-response analysis. Curve 1 of Figure 4 is the Nyquist diagram for G, with a change in scale factor to compensate for the action of the sampling device. By a simple process of vector addition, this curve is modified graphically to account for a finite number of complementary signals. Because of the periodicity and the symmetry of the function being approximated, the frequency-response curves need be plotted only over a range of frequencies equal to half of the sampling frequency. Only the first two terms of the infinite series are plotted in curve 2 of Figure 4. Curve 3, based upon the first two terms of the series, represents the same system with a lower sampling rate.

The Bode Diagrams corresponding to curves 2 and 3 in Figure 4 are curves 1 and 2, respectively, in Figure 5; and the corresponding Nichols curves are curves 1 and 2, respectively, in Figure 6. It is worth noting that the question of absolute stability can be answered by application of the conventional rules to the curves in Figures 4, 5, and 6. It should also be noted that constant-M contours lose their significance when sampling is involved.

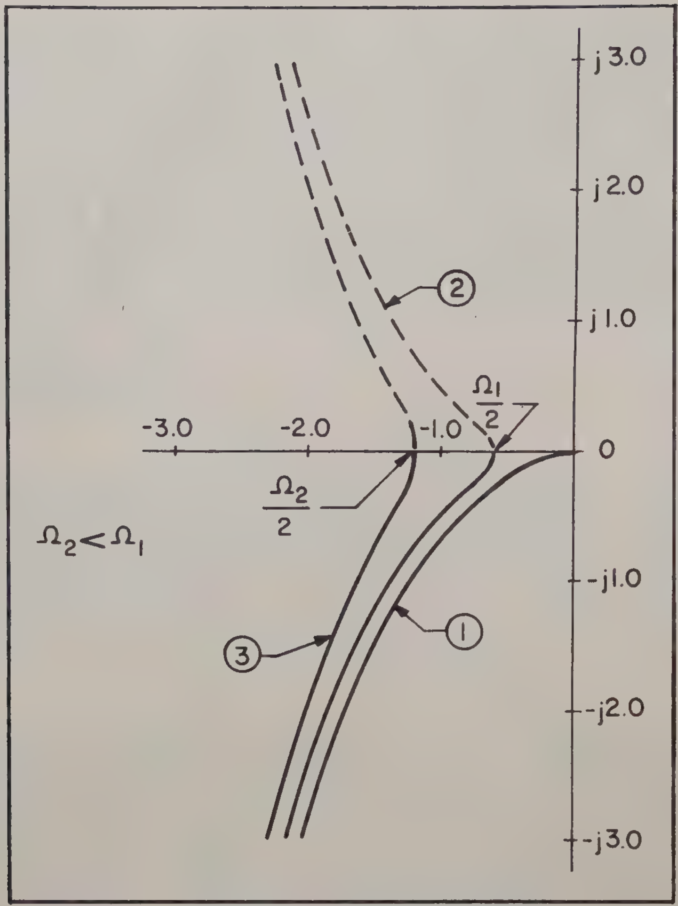


Fig. 4 - Nyquist diagrams for the system of figure 2.

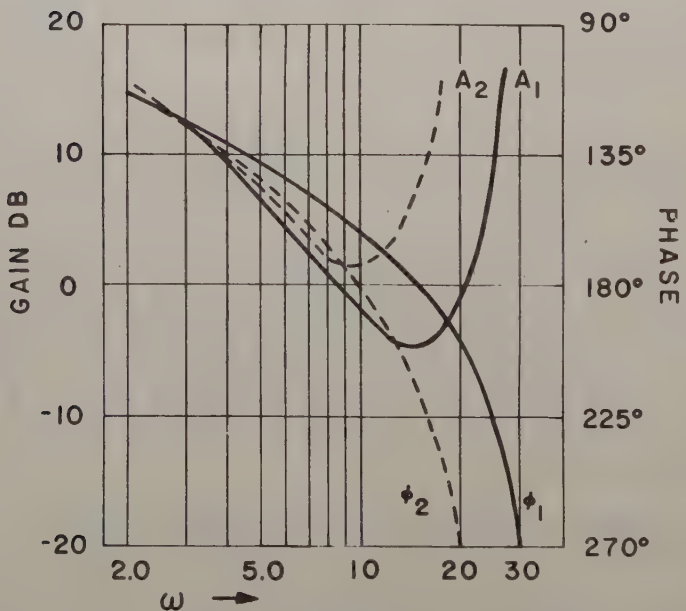


Fig. 5 - Bode diagrams for the system of figure 2.

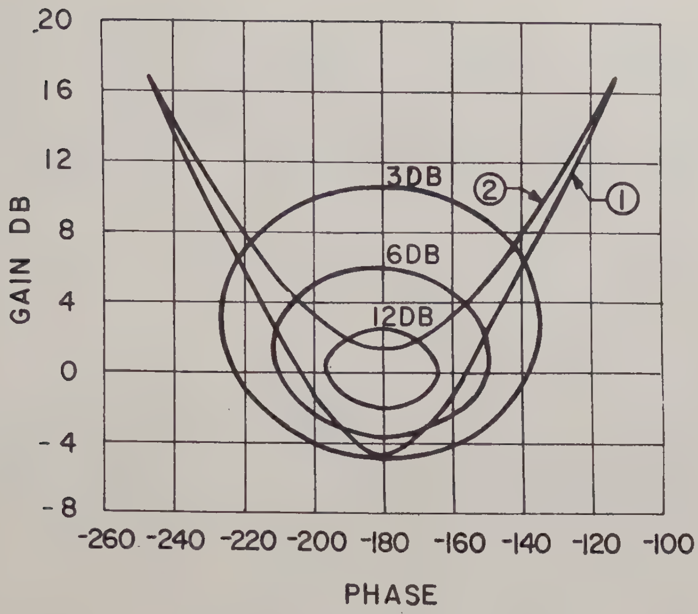


Fig. 6 - Nichols charts for the system of figure 2.

However, a limited amount of information concerning relative stability is contained in the nearness of the frequency-response curve to the critical point, i.e., (-1, 0) in the KG-plane.

The Z Transformation

Frequency-response techniques lead to a rapid approximate picture of the behavior of the sampled-data system. When more accurate information is required and the system is too complex for convenient use of the impulse-response analysis, it is advisable to write the exact transfer function for the sampled-data system in closed form and apply the techniques of transfer-function analysis. Because of the sampling, the transfer function is not a rational function of the Laplace transform variable s; as a result, the use of conventional techniques in the s-plane is ruled out. It is possible, however, to introduce a transformation,[10] replacing e^{sT} with z, where T is the period of the sampling device. This transformation results in a transfer function which is a rational function of z. Consequently, analysis on the basis of pole-zero configurations in the z-plane for sampled-data systems is similar to analysis on the basis of pole-zero configurations in the s-plane for systems which do not contain sampling devices.

The z-transform analysis procedure suffers from two limitations: (1) a new concept must be mastered by the engineer before proceeding with the system analysis: (2) without modification, the z-transform approach provides information about the signals at the sampling instants only, as does the frequency response method. It is possible to modify the basic method to obtain the value of a given signal at other than the sampling moments, but this modification requires considerable additional work.

The weight placed upon these limitations depends primarily upon the application. The engineer who spends much time working with sampled-data

systems will find it greatly to his advantage to master the convenient and circumspective z-transform techniques. The lack of information between sampling instants is not normally a serious handicap; the signal is usually sufficiently smooth to insure satisfactory behavior between sampling instants when the response at the sampling instants is satisfactory. The following example is included to illustrate the method of obtaining the z-transform of a known function of time.

For the function

$$f(t) = e^{-at}$$

with sampling assumed to begin at $t = 0$, the magnitude of the first sample is unity; that of the second sample is e^{-aT}; that of the third sample is e^{-2aT}; etc. By associating the magnitude of the sample at a given instant with the magnitude of an impulse assumed to occur at that instant, it is possible to write:

$$f^*(t) = u_o(t) + e^{-aT} u_o(t - T) + e^{-2aT} u_o(t - 2T) + \dots\dots$$

The Laplace transform of this signal is

$$Lf^*(t) = F^*(s) = 1 + e^{-aT} e^{-sT} + e^{-2aT} e^{-2sT} + \dots\dots$$

$$= \frac{1}{1 - e^{-aT} e^{-sT}}$$

Replacing e^{+sT} with z then gives a function which is said to be the z-transform of $f(t)$ and which for simplicity of notation is denoted with $F(z)$. Thus,

$$F(z) = \frac{z}{z - e^{-aT}}$$

Note that the z-transform is the same for all functions of time which are equal to one another at each sampling instant, even though these functions may differ greatly from one another between sampling moments.

The transfer function of a sampled-data device is defined to be the ratio of the transform of the sampled output to the transform of the corresponding sampled input. For the system in Figure 7, if the input r is a pulse at $t = 0$ of unit height, with duration less than T, the output c is e^{-at}. The z-transform of e^{-at} has been shown above to be $z/(z - e^{-aT})$, and the z-transform of the unit-pulse input is obviously unity. Hence, the transfer function of the system in Figure 7 is

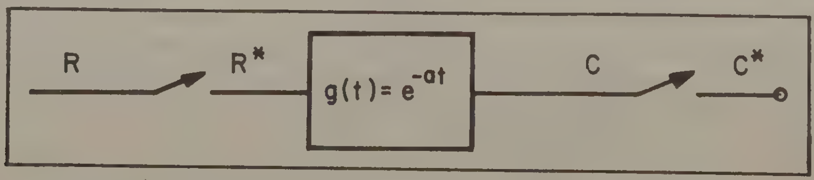


Fig. 7 - Illustration of the significance of the Z-transformed system function.

$$\frac{C(z)}{R(z)} = \frac{z}{z - e^{-aT}}$$

The stability analysis of a closed-loop sampled-data system such as that in Figure 1 may be carried out in terms of the Nyquist diagram for $C(z)/E(z)$ by plotting this complex quantity for values of $z = e^{jwT}$. (Such a curve is actually the exact curve which was approximated with a finite number of terms in the so-called frequency-response method discussed previously.) However, as in the study of continuous-data systems, much more information about the performance of the closed-loop sampled-data system can be obtained by plotting root loci in the z-plane and correlating the pole-zero constellation of the closed-loop system with its transient response.

The choice of compensation for sampled-data control systems can be made on the basis of the reshaping of the root locus in the z-plane. The use of a digital computer as a compensating device results in more flexibility in compensation than is possible with passive networks and permits the design procedure to be carried out completely in the z-plane.

The block diagram of a simple system which includes a digital computer is shown in Figure 8. The closed-loop transfer function is given by the expression

$$\frac{C}{R}(z) = \frac{D(z)\,G(z)}{1 + D(z)\,G(z)}$$

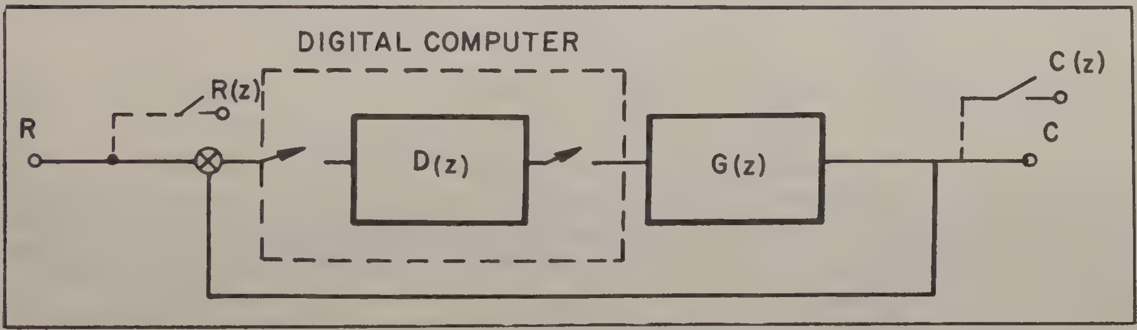


Fig. 8 - A simple feedback control system which incorporates a digital computer.

The fact that the digital program and the continuous part of the system are isolated by samplers permits the multiplication of the individual transfer functions in the z-domain. The ease with which compensation of continuous systems may be designed in the s-plane stems from the ability to multiply the transfer functions (the poles and zeros of the combined transfer function comprise all of the poles and zeros, respectively, of the individual transfer functions).

The design of digital programs for the shaping of the root locus in the z-plane is actually simpler than the design of compensation networks for continuous systems. Compensation for continuous systems is commonly obtained from networks composed of resistors and capacitors; the realization of such networks is restricted by the requirements that poles be simple and lie on

the negative real axis, and that the gain be limited by the response characteristics desired. The only requirement for realizability of a digital program is that it be expressible as a rational algebraic function of the delay operator z, and that b_o of the expression below be other than zero. The transfer function is of the form

$$D(z) = \frac{a_o + a_1(1/z) + a_2(1/z^2) + a_3(1/z^3) + \ldots + a_m(1/z^m)}{b_o + b_1(1/z) + b_2(1/z^2) + b_3(1/z^3) + \ldots + b_n(1/z^n)}$$

where the a's and b's are real coefficients.

The transfer function $D(z)$ may be expressed as $0(z)/I(z)$ where $0(z)$ is the computer output and $I(z)$ the computer input. The equation above may be written as

$$0(z)\left[b_o + b_1\frac{1}{z} + b_2\frac{1}{z^2} + b_3\frac{1}{z^3} + \ldots + b_n\frac{1}{z^n}\right] = I(z)\left[a_o + a_1\frac{1}{z} + a_2\frac{1}{z^2} + a_3\frac{1}{z^3} + \ldots + a_m\frac{1}{z^m}\right]$$

Choosing the a's and b's so that b_o becomes unity and rearranging give

$$0(z) = I(z)\left[a'_o + a'_1\frac{1}{z} + a'_2\frac{1}{z^2} + \ldots + a'_m\frac{1}{z^m}\right] - 0(z)\left[b'_1\frac{1}{z} + b'_2\frac{1}{z^2} + \ldots + b'_n\frac{1}{z^n}\right]$$

The power of z distinguishes the sample, and the a's and b's represent the weighting of the sample. For example, $I(z)\,a_2\,(1/z^2)$ specifies that the input two sampling periods earlier should be multiplied by a_2. If b_o had been zero, the program would have called for a future value of input and thus become unrealizable.

Discussion

Of the three most prominent methods of analyzing sampled-data control systems, the impulse-response analysis is the most straightforward. Moreover, it provides the maximum information concerning the system response to an arbitrary input. Because the application of this method is laborious, it is not practical for analysis of the more complex systems.

A method of analysis which is much simpler to apply but which yields less information is the so-called frequency-response method, which consists of a graphical approximation of the Nyquist diagram for the sampled-data system. When the only information required pertains to absolute stability or rough measurement of relative stability, the frequency-response analysis is rapid and adequate. It is also useful in design work by giving an indication of the type of compensation required. For a detailed analysis of the more com-

plex systems, however, transfer-function analysis in terms of the z-transform operator is the best of the three methods; it provides more information than the frequency-response analysis and is less laborious to apply than the impulse-response analysis.

The three methods discussed are not the only methods which have been proposed, but they appear to be the most widely accepted. A familiarity with the details of these three techniques should enable one to read, understand, and apply the results of forthcoming papers on the subject of sampled-data feedback-control systems.

Acknowledgment

This paper is based upon work performed for the Aeronautics Division of the Minneapolis-Honeywell Regulator Company. The authors wish to acknowledge the cooperation of Professor J. R. Ragazzini of Columbia University in making available numerous reports dealing with the z-transform.

REFERENCES

1. Fundamental Theory of Servomechanisms, Leroy A. MacColl. D. Van Nostrand Company, New York, 1945.

2. Theory of Servomechanisms, James, Nichols, and Phillips. McGraw-Hill, New York, 1947.

3. "A List of Generalized Laplace Transforms," W. M. Stone. Journal of Science Iowa State College, Ames, Iowa, vol. 22, April 1948, pp. 215-225.

4. "The Theory of Pulse Monitored Servomechanisms and Their Use for Prediction," R. H. Barker, S.R.D.E. Report No. 1046, Christ Church, Hunts, England, November 1950.

5. "Analysis of a Sampling Servomechanism," K. S. Miller and R. J. Schwarz. Journal of Applied Physics, vol. 21, no. 4, April 1950, pp. 290-294.

6. "Sampled-Data Control Systems Studied Through Comparison of Sampling with Amplitude Modulation," William K. Linvill. Transactions, AIEE, 1951.

7. "A Treatment of Digital Control Systems and Numerical Processes in the Frequency Domain, "J. M. Salzer. Sc.D. thesis, Massachusetts Institute of Technology, 1951.

8. "A General Theory of Sampling Servo Systems," D. F. Lawden. Monograph, IEE, no. 4, 1951.

9. Automatic and Manual Control, Tustin. Butterworth's Scientific Publications, London, 1952.

10. "The Analysis of Sampled-Data Systems," J. R. Ragazzini and L. A. Zadeh. Transactions, AIEE, 1951.

11. "An Approximate Transfer Function for the Analysis and Design of Pulsed Servos," R. G. Brown and G. J. Murphy. Transactions, AIEE, 1952.

12. "The Pulse Transfer Function and Its Application to Sampling Servo Systems," R.H. Barker, Proceedings, IEE, London, England, vol. 99, part IV, December 1952, pp. 302-317.

13. "Analysis and Synthesis of Sampled-Data Systems," E. I. Jury. Eng. Sc.D. thesis, Columbia University, 1953.

14. "Design of Sampled-Data Systems by Extension of Conventional Techniques," W. K. Linvill and R. W. Sittler. Digital Computer Laboratory Report R-222, MIT, July, 1953.

15. "Extension of Conventional Techniques to the Design of Sampled-Data Systems," W. K. Linvill and R. W. Sittler. Convention Record, IRE, part I, 1953, pp. 99-104.

16. "Analysis of Control Systems Involving Digital Computers," W. K. Linvill and J. M. Salzer. Proceedings, IRE, July 1953, pp 901-906.

17. "The Design of Sampled-Data Feedback Systems," G. V. Lago and J. G. Truxal. Transactions, AIEE, 1954.

18. "Analysis and Synthesis of Sampled-Data Control Systems," E. I. Jury. Transactions, AIEE, 1954.

19. "Sampled-Data Processing Techniques for Feedback Control Systems," A. R. Bergen and J. R. Ragazinni, Transactions, AIEE, 1954.

20. "The Analysis and Design of Sampled-Data Systems," G. V. Lago. Ph.D. thesis, Purdue University, May 1954.

21. "Additions to Z-Transform Theory for Sampled-Data Systems," G. V. Lago. AIEE Paper No. 54-523.

22. "The Frequency Analysis of Digital Computers Operating in Real Time," J. M. Salzer. Proceedings, IRE, 1954.

23. "Linear Filtering of Sampled Data," Gene F. Franklin. Electronics Research Laboratories, Columbia University, Technical Report T-5/B, December 1954.

24. "Contactor Servomechanisms Employing Sampled Data," C. K. Chow. Transactions, AIEE, 1954, pp. 51-62.

25. "A Low-Pass Transformation for Z-Transforms," (correspondence), W. H. Huggins. Transactions, IRE, September 1954, pp 69-70.

26. "Transient Analysis of Sampled-Data Control Systems," C. W. Johnson and D. P. Lindorff. Transactions, AIEE, July 1954, pp. 147-153.

27. "Analysis of Errors in Sampled-Data Feedback Systems," J. Sklansky and J. R. Ragazinni, AIEE Paper No. 55-189.

28. "The Effect of Pole and Zero Locations on the Transient Response of Sampled-Data Systems," E. I. Jury. AIEE Paper No. 55-186.

29. Unpublished class notes, J. R. Ragazinni. Course EE 208, Columbia University.

30. "The Optimum Synthesis of Sampled-Data Systems," Gene F. Franklin. Electronics Research Laboratories, Columbia University, Technical Report T-6/B, May 2, 1955.

31. "Data Smoothing Techniques Final Report," P. H. Rudy, A. J. Mallinchrodt, E. I. Bosch, and G. F. McCormick. Ralph M. Parsons Company, Pasadena, California.

32. "Network Compensation of Error-Sampled Feedback Control Systems," J. Sklansky. Electronics Research Laboratories, Columbia University, Technical Report T-7/B, April 1, 1955.

33. Automatic Feedback Control Synthesis, J. G. Truxal. McGraw-Hill, New York, N. Y., 1955.

34. "Closed-Loop Control System Containing a Digital Computer," T. Teichmann. IRE Transactions on Electronic Computers, vol. EC-4 September, 1955.

35. "Extension of Continuous-Data System Design Techniques to Sampled-Data Control Systems," C. W. Johnson, D. P. Nordling, D. P. Lindorff. Transactions, AIEE, September, 1955, pp 252-263.

36. "Multipole Sampled-Data Control Systems, H. Freeman, Electronics Research Laboratories, Columbia University, Technical Report T-12/B, September 30, 1955.

37. "A Technique for the Analysis of Time-Varying Sampled-Data Systems," B. Friedland, Electronics Research Laboratories, Columbia University, Technical Report T-10/B, September 15, 1955.

38. "Digital Compensation of Continuous-Data Feedback Control Systems," K. K. Maitra, P. E. Sarachik. AIEE Paper No. 56-210.

39. "Factors in the Design of Digital Controllers for Sampled-Data Feedback Systems," J. E. Bertram. AIEE Paper No. 56-209.

40. "Synthesis and Critical Study of Sampled-Data Control Systems," E. I. Jury, AIEE Paper No. 56-208.

41. "Correlation Between Root-Locus and Transient Response of Sampled-Data Control Systems," E. I. Jury. Transactions, AIEE, January, 1956, pp. 427-435.

42. "Transformation Techniques for Time-Varying Sampled-Data Systems," Bernard Friedland, Electronic Research Laboratories, Columbia University, Technical Report T-13/B, January 2, 1956.

43. "Nonlinear Aspects of Sampled-Data Control Systems," R. E. Kalman. Proceedings of the Symposium on Nonlinear Circuit Analysis (MRI Symposia Series Volume VI) Polytechnic Institute of Brooklyn, 1956.

IRE Standards on Graphical and Letter Symbols for Feedback Control Systems, 1955*

I. Introduction

1.1 Standardization of symbols is considered important for the exposition of feedback control concepts. The purpose is to establish forms for representing letter symbols and graphical symbols used in block diagrams.

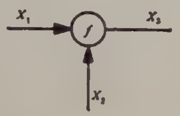
In the preparation of this standard, it was found that there was no complete and self-consistent set of symbols in use that appeared to fulfill the requirements of the IRE. It was also found that there is a wide variation in the symbols used by different industries and professional societies so that it is difficult to choose symbols with a universal acceptance in all fields. Therefore, a compromise has been made among existing symbols used in the electrical field.

II. Graphical Symbols for Block Diagrams

2.1 *Transfer Element*

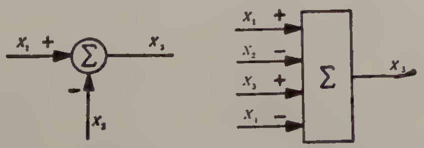
2.1.1 A transfer element represents the functional relationships (g_{12}) between a single input signal (x_1) and a single output signal (x_2), in which the input signal, indicated by the arrow, is the independent variable.

2.2 *Mixing Point*



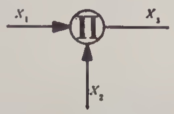
2.2.1 The indicated relationship is $x_3 = f(x_1, x_2)$.

2.3 *Summing Point*



2.3.1 The indicated relationships are $x_3 = x_1 - x_2$ and $x_5 = x_1 - x_2 + x_3 - x_4$. A summing point is a special case of the mixing point and indicates the algebraic addition of two or more signals to produce one output signal. An algebraic sign should be indicated at the arrowhead for each signal to be added. If the number of input signals to be added is large the rectangular symbol should be used.

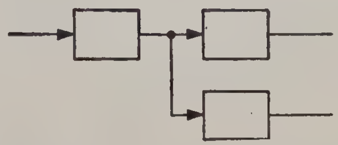
2.4 *Multiplication Point*



2.4.1 The indicated relationship is $x_3 = x_1 x_2$. A multiplication point is a special case of the mixing point.

2.5 *Branch Point*

2.5.1 A branch point, which indicates that a signal is distributed to two or more points in a block diagram is represented by a heavy dot. Example:



2.6 Graphical symbols added to a block diagram for mathematical purposes shall be shown dotted to indicate that they do not represent components of the physical system.

III. Standard for Letter Symbols

3.1 *Essential Features of the System of Symbols*

3.1.1 *Signals*

Signals are represented by a single letter symbol with a single subscript denoting its physical or mathematical meaning. The letter x has been chosen as the preferred symbol for generalized signals. Lower case represents the time domain. Upper case represents the complex frequency domain.

3.2 *Transfer Functions*

Transfer functions are represented by a single letter symbol with a double subscript, the first letter or number of which is the subscript of the symbol for the input signal and the second of which is the subscript of the symbol for the output signal. The symbol g has been chosen. Lower case represents the time domain. Upper case represents the complex frequency domain.

IV. Examples

4.1 Application of the standard graphical symbols and

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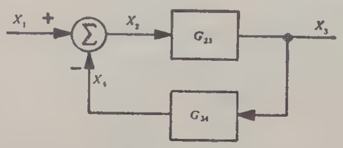
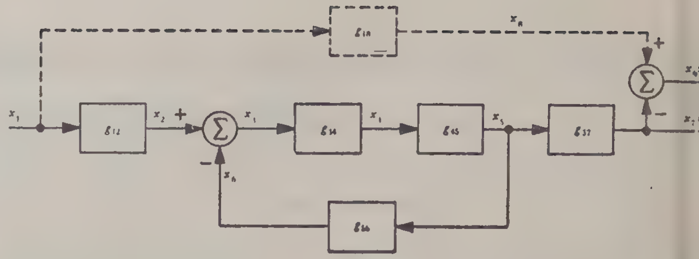
*Reprinted from the Proceedings of the IRE, November, 1955.

the standard form for letter symbols are illustrated in the typical block diagrams below:





IRE Standards on Terminology for Feedback Control Systems, 1955*

1. Introduction

1.1. Prior to the preparation of this standard, no complete and self-consistent set of terminology appeared to fulfill the requirements of the IRE. Because of a wide variation in the terminology used by different industries and professional societies it was difficult to choose a terminology acceptable to all fields. Therefore, a compromise has been made among existing terminologies used in the electrical field.

2. Definitions

2.1 *Signals*

2.1.1. **Loop Input Signal.** An external signal applied to a feedback control loop.

2.1.2. **Loop Output Signal.** The controlled signal extracted from a feedback control loop.

2.1.3. **Loop Feedback Signal.** The signal derived as a function of the loop output signal and fed back to the mixing point for control purposes.

2.1.4. **Loop Actuating Signal.** The signal derived by mixing the loop input signal and the loop feedback signal.

2.1.5. **Loop Error Signal.** The loop actuating signal in those cases in which it is the loop error (See 2.4.10).

2.1.6. **Loop Return Signal.** The signal returned via a feedback control loop to a summing point, in response to a loop input signal applied to that summing point, and subtracting from the loop input signal.

Note: The loop return signal is a specific type of loop input signal.

2.1.7. **Loop Difference Signal.** The output signal from a summing point of a feedback control loop produced by a particular loop input signal applied to that summing point.

Note: The loop difference signal is a specific type of loop actuating signal.

2.2. *Points and Paths*

2.2.1. **Mixing Point.** In a block diagram of a feedback control loop, a symbol indicating the relationship of one output to two or more inputs, such that the value of the output at any instant is a function of the values of the inputs at that instant.

Note: If a mixing device in practice contains dynamic elements, these shall be considered as transfer elements in one or more of the signal paths entering or leaving the mixing point.

2.2.2. **Summing Point.** A mixing point whose output is obtained by addition, with prescribed signs, of its inputs.

2.2.3. **Multiplication Point.** A mixing point whose output is obtained by multiplication of its inputs.

2.2.4. **Forward Path.** In a feedback control loop, the transmission path from the loop actuating signal to the loop output signal.

2.2.5. **Feedback Path.** In a feedback control loop, the transmission path from the loop output signal to the loop feedback signal.

2.2.6. **Through Path.** In a feedback control loop, the transmission path from the loop input signal to the loop output signal.

2.3. *Transfer Functions*

2.3.1. **Transfer Function.** A relationship between one system variable and another that enables the second variable to be determined from the first.

2.3.2. **Transfer Ratio.** The transfer function from one system variable to another in a linear system, expressed as the ratio of the Laplace transform of the second variable to the Laplace transform of the first variable, assuming zero initial conditions.

*Reprinted from the Proceedings of the IRE, January, 1956.

2.3.3. **Loop Transfer Function.** The transfer function of the transmission path formed by opening and properly terminating a feedback loop.

Note: One example of proper termination is a zero impedance generator driving the opened loop, and an output termination for the opened loop equal to the impedance facing the generator.

2.3.4. **Loop Transfer Ratio.** The transfer ratio of a loop return signal to the corresponding loop difference signal.

2.3.5. **Forward Transfer Function.** In a feedback control loop, the transfer function of the forward path.

2.3.6. **Feedback Transfer Function.** In a feedback control loop, the transfer function of the feedback path.

2.3.7. **Return Transfer Function.** In a feedback control loop, the transfer function which relates a loop return signal to the corresponding loop input signal.

2.3.8. **Difference Transfer Function.** In a feedback control loop, the transfer function which relates a loop difference signal to the corresponding loop input signal.

2.3.9. **Through Transfer Function.** In a feedback control loop, the transfer function of the through path.

2.3.10. **Actuating Transfer Function.** In a feedback control loop, the transfer function which relates a loop actuating signal to the corresponding loop input signal.

2.4. *General*

2.4.1. **Feedback Control Loop.** A closed transmission path, which includes an active transducer and which consists of a forward path, a feedback path, and one or more mixing points arranged to maintain a prescribed relationship between the loop input signal and the loop output signal.

2.4.2. **Feedback Control System.** A control system, comprising one or more feedback control loops, which combines functions of the controlled signals with functions of the commands to tend to maintain prescribed relationships between the commands and the controlled signals.

2.4.3. **Feedback Control System, Linear.** A feedback control system in which the relationships between the pertinent measures of the system signals are linear.

2.4.4. **Feedback Control System, Nonlinear.** A feedback control system in which the relationships between the pertinent measures of the system input and output signals cannot be adequately described by linear means.

Note: A system can be either *quasi-linear* or *nonlinear*, depending upon operating conditions and performance requirements.

2.4.5. **Feedback Control System, Quasi-Linear.** A feedback control system in which the relationships between the pertinent measures of the system input and output signals are substantially linear despite the existence of nonlinear elements.

Note: A system can be either *quasi-linear* or *nonlinear*, depending upon operating conditions and performance requirements.

2.4.6. **Feedback Regulator.** A feedback control system which tends to maintain a prescribed relationship between certain system signals and other pre-determined quantities.

Note 1: This definition is intended to point out the fact that some of the system signals in a regulator are adjustable reference signals.

Note 2: It should be noted that *servomechanism* and *regulator* are not mutually exclusive terms; their application to a particular system will depend on the method of operation of that system.

2.4.7. **Servomechanism.** A feedback control system in which one or more of the system signals represent mechanical motion.

Note: It should be noted that *servomechanism* and *regulator* are not mutually exclusive terms; their application to a particular system will depend on the method of operation of that system.

2.4.8. **Command.** An independent signal from which the dependent signals are controlled according to the prescribed system relationships.

2.4.9. **Disturbance.** An undesired command.

2.4.10. **Loop Error.** The desired value minus the actual value of the loop output signal.

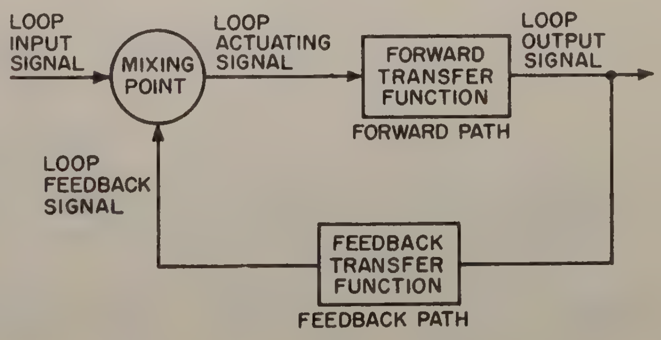


Fig. 1

3. Example

3.1. Application of the standard terminology to a typical block diagram is shown in Fig. 1.

LETTER TO THE EDITOR

On Terminology for Feedback Control Systems

What's all this fuss about terminology for feedback control systems? It is well known that when we attempt to design an amplifier an oscillator results. Conversely, when an oscillator is desired a highly stable amplifier is realized. Why not put this principle to work? A design method for servos is described below which makes maximum utility of the above lemma. The approach is dependent on the development of a set of inconsistent, contradictory, inherently unstable terms to make the servo system believe it is intended for use as an oscillator. Such a set of terms is appended to this note. The design procedure then becomes a three-step process.

1. Show the following daffinitions to the system under design.

2. Invoke the "Law of the Perversity of Inanimate Objects." We have found that this procedure invariably results in a "sailable" product which far exceeds our performance spec's. For those people who have little faith in this approach we recommend the third step.

3. Test the system on a psychiatrist's couch (with or without the doubting engineer). This simple test can be accomplished without a flight table and uses the bare minimum of test equipment.

In passing, it should be noted that our approach has been extremely successful in designing feedback control loops containing a human operator. The exact origin of the attached definitions is unknown. However, when my sister and I are released from our present sanctuary we intend to offer a well earned reward to the originator of these terms.

Yours truly,

Mr. and Miss Informed
Flying Saucerville
In Space

Daffinitions

Servomechanism: A system, usually of great complexity, in which the output of a device is sampled and fed back into the input in order to produce uncontrollable oscillations.

Feedback loop: That portion of a servomechanism circuit which makes possible its instability.

Follow-up: A device in a servomechanism which is used to follow, as faithfully as possible, its oscillations.

Rate-generator: A small but costly electrical device affixed to the output end of a servo system for ornamental purposes.

Stability: The desired optimum of servo performance, as manifested by violent thrashing of an output member.

Amplifier: An electronic device for superimposing electrical instabilities on mechanical.

Zero back-lash: A whimsical concept much professed by authors of specifications.

Specification: A form of heroic or epic poetry favored by engineers; a collection of impossible conditions; a collection of loop-holes loosely held together by wistful verbiage.

Wiring-diagram: A form of the graphic arts which has the characteristic of changing form constantly in the manner of cloud formations.

Nyquist Diagram: An ornamental figure derived from mathematics which demonstrates the stability of an obviously unstable system.

Cathode-ray oscilloscope: A television set devoid of advertizing or wrestling matches.

Production (with respect to servo electronics): The process of painfully assembling, dismantling, and reassembling electrical components in accordance with wiring-diagrams (q.v.); a vicious circle.

Servomechanisms engineer: A soothsayer of mystical powers skilled in mathematical rubrics and incantations; a prophet without honor in California or anywhere else; a dreamer or woolgatherer.

Performance evaluation: A grateful appreciation of whatever results from a servo system.

ABSTRACTS OF MINUTES OF THE MEETING OF THE ADMINISTRATIVE COMMITTEE OF THE PROFESSIONAL GROUP ON AUTOMATIC CONTROL, INSTITUTE OF RADIO ENGINEERS, HELD AT IRE HEADQUARTERS, SEPTEMBER 17, 1956

1. The meeting was called to order at 10:20 a.m.

2. PGAC Technical Sessions at IRE 1957 National Convention: Meetings Committee Chairman Grabbe has been appointed to represent the Group on the 1957 IRE National Convention Technical Program Committee. Mr. David Lindorff will serve as his alternate.

   In addition to the papers received at Headquarters in answer to the "call for papers" for the Convention, Messrs. Grabbe and Lindorff will solicit some invited papers for the PGAC Technical Sessions. It was suggested that the residue of papers submitted be considered for future use at other meetings and that they also be referred to Editor Axelby for consideration for the *Transactions*.

   Mr. Grabbe will report to the next meeting of the Committee on the number of technical sessions which will be allotted to the Group by the Convention Technical Program Committee and the number and quality of the papers received.

3. Committee Organization: The chairman reported on his proposed plans for the coming year and made the following Committee appointments:

   3a. Awards Committee: Mr. George Biernson was reappointed Chairman of this Committee.

   3b. Meetings Committee: Mr. Eugene Grabbe will serve as Chairman of this Committee. Mr. David Lindorff will assist him. Mr. Grabbe was authorized to appoint other members to the Committee as required to procure papers for all meetings.

   3c. Membership Committee: Mr. Azgapetian will continue as Temporary Chairman of the Membership Committee until his successor is appointed.

   This Committee will handle all of the following activities: membership promotion, chapter promotion, publicity, and student activities. Chairman Lozier will appoint four members to handle each of the above-named activities, and a new Membership Committee Chairman to coordinate these activities.

   3cl. It was moved that the sum of $250.00 be allotted from the Group Treasury for a supply of PGAC Membership Brochures. Mr. Azgapetian will prepare the text for this brochure and submit it to the Chairman and Secretary for approval by October 17th (unanimous). These brochures will be used for membership promotion at the IRE National Convention, WESCON, NEC and the Annual Meeting of the Group; for distribution to Chapters for their membership activities; and for membership promotion mailings initiated by the Membership Committee.

3c2. Chapters: Mr. Wilcox agreed to accept the Chairmanship of the Membership Subcommittee on Chapters. He also accepted the responsibility for preparing copy for a letter that can be used by a Chapter organizer to assist in the formation of a Chapter.

3c3. Publicity: The primary function of this Subcommittee would be to anticipate the necessary publicity for "calls for papers," to publicize proposed Chapter activities and interesting features of National and Chapter meetings. This would include dissemination of information to the Press and magazines. Mr. Aaron agreed to accept the Chairmanship of this Membership Subcommittee on Publicity.

All Chapter Chairmen are hereby requested to send information that can be used for publicity to Mr. Aaron. A copy of papers of interest to the profession in particular and to the public in general, should be forwarded in advance of their presentation with their date of presentation. This will enable releases to technical magazines and the Press. Releases after the presentation are almost never accepted.

3d. Papers Procurement and Publications Committee: It was proposed that Editor Axelby chairman the activities of a combined papers study, papers procurement, papers review and publications committee. Mr. Axelby was authorized to appoint such other members to this Committee as may be necessary to carry on its activities. This should include the appointment of one person who would be responsible for coordination with other Professional Groups. It is hoped that the Abstracts and Reviews Section of the Transactions can be handled on an international basis.

3e. It was the concensus of opinion that the Group sponsor an annual symposium along the lines suggested by Mr. Axelby as soon as this is practicable. After a lengthy discussion it was decided to table this point until the next meeting at which time a Symposium Committee will be appointed.

4. Chapter Reports:

4a. Los Angeles Chapter: Mr. Grabbe reported that the Los Angeles Chapter had succeeded in increasing its meeting attendance by presenting papers on new advances.

4b. Twin Cities Chapter: Chapter Chairman La Hue reported that the Twin Cities Chapter had enjoyed very good attendance at its meeting on "Inertial Navigation."

4c. Akron Chapter: It was moved that the petition presented to Headquarters for the formation of an Akron Chapter be approved (unanimous).

4d. New York, Long Island and Northern New Jersey: An effort will be made to contact organizers in these Sections in order to form Chapters.

4e. Baltimore Chapter: Mr. Axelby reported that two meetings have been planned for the coming year and the papers presented may be published in the Transactions. The AIEE Baltimore Section has approached the PGAC Baltimore Chapter with the suggestion that joint meetings be held.

5. Membership Survey: It was decided to prepare and distribute a questionnaire to the Membership to determine which of the members are interested in doing Committee work, either on a local Chapter level or on a National level, or for symposia or papers procurement and review. This questionnaire should also attempt to determine what type of programs, meetings and papers for publication would serve the membership in the best manner. The Chairman and Secretary will prepare the text for this questionnaire. It was suggested that in some cases it would be desirable to address letters to Management to request their assistance in allowing time for engineers to work on PGAC projects. This approach will be followed after the survey has been completed.

6. Finance Report: The balance in the Group Treasury on August 31, 1956, was $4,239.05.

7. Membership: As of August 31, 1956, membership was as follows:

Paid	1,482
Paid Students	157
Unpaid	159
Total	1,798

8. Date of Next Meeting: The next meeting will be held on Monday, December 3, 1956, at 10:00 a.m. at IRE Headquarters, New York.

9. There being no further business, the meeting adjourned at 3:30 p.m.

M. Robert Aaron
Secretary-Treasurer

Issued: September 28, 1956

IRE PROFESSIONAL GROUP ON AUTOMATIC CONTROL
MEMBERSHIP DIRECTORY (AS OF 5 NOVEMBER 1956)

REGION I

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Bosman, E. H.
Cheng, Tsung-Hsien
Hamburgen, Arthur
Hunt, J. M.
Ivy, R. C.
Kilmer, F. G.
Lavender, R. W.
Shatz, J. R.
Tutty, J. E.

Boston Section

Alden, J. M.
Allen, Jonathan
Anderegg, J. S.
Antul, J. J.
Applegate, C. E.
Barabush, Arthur
Barry, J. G.
Batchelder, Laurence
Beaudette, C. G.
Bell, C. G.
Benkley, F. G.
Bennett, H. W.
Bennett, R. K.
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Blanchard, R. L.
Blanton, H. E.
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Brew, R. A.
Brooks, P. R., Jr.
Brown, G. S.
Brown, J. B.
Bullard, A. H., Jr.
Burgess, A. G.
Burwen, R. S.
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Freudberg, R. L.
Fricks, R. E.
Galagan, Steven
Gallagher, E. F.
Galluzzi, N. P.
Gitelman, Ephraim
Goldberg, David
Gordon, B. M.
Goulder, M. E.
Grossimon, H. P.
Hawkins, R. S.
Heaviside, M. G.
Heuchling, T. P.
Hillman, H. D.
Hills, F. B.
Hollins, C. G.
Hoogasian, Leon
Horth, T. C.
Howard, R. A.
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Hulburt, H. J., Jr.
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Hynek, D. P.
Iffland, J. J.
Johnson, E. P., Jr.
Kasvand, Tonis
Klemperer, Hans
Krulee, R. L.
Leath, C. W.
Lechner, R. J.
Leonard, C. E.
Leonard, R. R.
Levy, D. M.
Luoma, R. A.
Lyden, J. A., Jr.
Mahoney, T. F.
Marino, A. S.
Markey, J. T.
Martin, L. H.
Martin, T. E.
Martin, T. H.
Meditch, J. S.
Melanson, F. J.
Mercer, W. R.
Merrill, Brian
Miles, R. J.
Minnick, R. C.
Minsky, M. L.
Missio, D. V.
Morell, C. S.
Morgenstern, J. C.
Morris, R. V.
Murano, Lodovico
Nagy, Ferenc, Jr.
Narendra, K. S.
Naylor, T. K.
Neidorf, Edward
Neilsen, C. E., Jr.
Oettinger, A. G.
Olsson, E. A.
O'Neil, S. J.
Orenberg, Arthur
Osman, M. S.
Palmer, P. J.
Pantazelos, P. G.
Pastan, H. L.
Pease, W. M.
Perez, A. A.
Perron, R. R.
Platt, H. J.
Prager, R. H.
Rittenburg, S. E.
Robinson, J. A.
Roch, M. E.
Rona, T. P.
Rosenthal, R. E.
Rowell, W. G.
Sabin, E. A.
Sanders, R. C., Jr.
Scanlon, W. C.
Schorr-Kon, J. J.
Seifert, W. W.
Senseman, R. W.
Shansky, David
Sheehan, J. A.
Sinclair, D. B.
Smith, R. L.
Soderstrom, R. E.
Solano, Joseph
Stabler, E. P.
Stockebrand, T. C.
Susskind, A. K.
Swonger, C. W.
Taylor, H. P.
Tsao, C. K. H.
Tunnicliffe, W. W.
Vacca, R. H.
Walsh, W. P.
Ward, J. E.
Wellons, R. S.
Wexler, H. T.
Whelan, W. J.
Whitcraft, W. A., Jr.
Whiteside, A. E.
Whittaker, H. F., Jr.
Wilcox, R. B.
Wilkie, L. E.
Williams, S. B.
Wilson, P. B., Jr.
Wing, Jack
Woodruff, T. E.
Woodward, J. H.
Yang, Chia-Chih
Zatlin, F. R.
Zieman, H. E.

Buffalo-Niagara Section

Aines, F. G.
Archibald, W. R.
Bachman, C. G.
Frank, L. E.
Green, V. H.
Grose, C. W.
Harriman, T. J.
Hayman, R. A.
Newton, D. J., Jr.
Novo, R. B.
Spoon, G. E.
Walbesser, W. J.

Connecticut Valley Section
Allen, T. F.
Bailey, E. M.
Bessette, D. U.
Bourret, C. J.
Brownell, R. M.
Curtiss, R. H.
Delany, E. B.
Duffy, J. F.
Earsy, R. M.
Edmondson, R. H.
Farwell, R. P.
Flynn, T. F.
Georgi, E. A., Jr.
Gibson, J. E.
Gilchrist, E. S.
Gray, N. L.
Haas, V. B., Jr.
Hall, B. A.
Harrington, C. F., Jr.
Helterline, L. L., Jr.
Jensen, H. T.
Kosowsky, L. H.
Lempert, Joseph
Lovell, B. W.
Mahar, T. F.
Mapes, T. J.
Martin, Devereaux
Miller, R. A.
Mills, R. H.
Norton, C. A.
Orell, F. M.
Page, J. F., Jr.
Paradise, R. Y.
Petrucelly, V. J.
Plehaty, S. L.
Poland, W. L.
Rathbun, W. F.
Saltarelli, M. O.
Segesman, F. F.
Stockman, W. E.
Sunega, E. M.
Sweet, R. D.
Turek, R. F.
Warhurst, J. S.
Weeks, J. T.
White, J. A.
Williams, J. B., Jr.
Zweig, Felix

Elmira-Corning Section
Attridge, C. C.

Ithaca Section
Clark, P. S.
Ittenbach, L. J., Jr.
November, P. M.

Rochester Section
Berch, W. H.
Brown, G. A.
Chesna, John
Dutcher, B. G.
Faust, A. C.
Formicola, A. F.
Maltais, W. E.
Mateyk, William
Merle, C. W.
Morse, J. E.
Sheehan, J. F.
Trott, Marvin

Rome-Utica Section
Ashcroft, D. L.
Boxer, Rubin
Glaser, G. J.
Hatfield, J. P.
Kabrisky, M. J.
Mayer, H. F.
McClure, R. W.
Ricci, B. E.
Ross, P. C.
Snider, G. L., Jr.
Walker, R. K.

Schenectady Section
Beattie, C. L.
Borner, E. F.
Buchhold, T. A.
Dabul, Amadeo
Dodson, G. C., Jr.
Gaines, W. M.
Keenan, P. P.
Lippitt, D. L.
McCarron, D. J.
Montgomery, E. B.
Muntz, W. E.
Rothe, F. S.
Schumacher, G. B.
Violette, J. L. N.
Ward, D. D., Jr.

Syracuse Section
Cottle, D. W.
Eisengrein, R. H.
Hellerman, Herbert
Johnson, G. L.
Jureller, J. F.
Kieburtz, R. B.
Mayo, B. R.
Neelands, L. J.
Rojak, F. A.
Russell, J. B., Jr.
Shuart, O. H.
Strong, W. J.
Vaughan, J. A.

REGION II

Long Island Section
Adise, H. H.
Amato, J. A.
Barbeau, A. R.
Barber, Edward
Behn, E. R.
Bergamo, M. V.
Burgess, E. G., Jr.
Cap, S. T.
Comenzo, L. F.
Corrado, V. M.
Costa, T. A.
Crosby, M. G.
Darden, R. R., Jr.
Dettinger, David
DiToro, M. J.
Dodd, F. L.
Doersam, C. H., Jr.
Dunning, O. M.
Duval, A. N.
Dyer, J. N.
Egnuss, E. M.
Epple, L. M.
Feay, D. I.
Ferrantino, S. J.
Firth, L. G., Jr.
Fishbein, Milton
Fonseca, A. P.
Frank, F. P. E.
Freeman, Herbert
Fried, George
Friedman, E. D.
Fromm, W. E.
Gitten, L. J.
Goodstein, Julian
Gretz, R. W.
Gross, S. H.
Hansen, H. R.
Harris, H. I.
Haynes, N. M.
Heacock, W. J., Jr.
Heimer, R. L.
Herman, Sidney
Hittner, Charles
Hoffer, L. H.
Jacob, G. W.
Joline, E. S.
Kay, L. M.
Kiel, J. H.
Klein, R. C.
Knocklein, H. P.
Knox, R. W.
Kokkinos, Constantinos
Lemanczyk, J. C.
Levinson, Emanuel
Lutz, C. H.
Marston, R. S.
Match, M. J.
McPherson, D. L., III
Meirowitz, R. L.
Mohr, H. F.
Moritz, F. G.
Morse, R. V.
Odessey, P. H.
Orford, R. J.
Osder, S. S.
Padovano, R. J.
Pagano, C. N.
Pearsall, C. H., Jr.
Peterson, H. O.
Pighi, L. H.
Prichard, J. S.
Purdy, R. L.
Rehberg, C. F.
Rosenthal, M. M.
Rosenthal, S. A.
Rubenson, J. G.
Sandock, Jules
Sayer, George
Scheideler, G. E.
Schneider, R. M.
Schoedel. R. A.
Schramm, M. W., Jr.
Schulkind, Donald
Scott, J. E.
Seckler, P. J. A.
Shanahan, W. J.
Sherry, L. I.

Sieminski, Edward
Simon, R. L.
Simonton, L. J.
Slusarchyk, G. J.
Smilowitz, S. N.
Steinberg, W. A.
Stephenson, J. G.
Stepper, L. H.
Tamasi, A. P.
Torn, L. J.
Varnum, A. M.
Vogel, Erwin
Walker, A. C.
Wathen, R. L.
Weintraub, Irving
Wellman, F. C.
Westover, T. A.
Wheeler, H. A.
Wiesner, Leo
Willis, P. A.
Winzemer, A. M.
Wolkov, David
Wurman, Gustave
Young, V. J.

New York Section

Albanese, A. P.
Antonazzi, F. J.
Arcidiacono, Joseph
Banner, Leonard
Barker, D. R.
Baumann, D. A.
Beeh, R. C. M.
Belluck, R. E.
Bertram, J. E.
Bibbero, R. J.
Blecher, Franklin
Blumberg, R. H.
Boyanovitch, Dushan
Boylan, L. K.
Brailey, M. L.
Buton, E. C., Jr.
Carpenter, E. J., Jr.
Clemens, G. J.
Connelly, J. J.
Cooper, P. C.
Cristina, A. A.
Cuccia, J. F.
Cypser, R. J.
D'Amato, R. J.
Dellenbaugh, F. S., III
DeWitt, R. G.
Dickstein, S. R.
Diebold, J. T.
Diehl, R. T.
Drescher, Charles
Duffy, J. J.
Ebaugh, D. P.
Evans, B. O.
Fabricant, B. S.
Feinerman, Bernard
Fernald, O. H.
Fisch, Alan
Foran, R. P.
Franklin, G. F.
Freudenberg, Boris
Friedensohn, George
Gardner, R. K.
Garman, R. L.
Gayet, P. J.
Georgallis, G. C.
Gilman, G. W.
Gister, Stanley
Glazer, Eli
Glowalla, John
Goldberg, Harold
Goldsmith, A. N.
Goldstein, Edward
Gronner, Alfred
Groudan, A. I.
Haddad, R. A.
Hamer, Howard
Heineck, A. W.
Hoekstra, Robert, Jr.
Hornung, S. A.
Horowitz, I. M.
Humphrey, R. M.
Jorysz, Alfred
Kalogeras, G. A.
Kaplan, K. R.
Karmel, P. R.
Kassel, Aaron
Katz, M. D.
Klein, W. T.
Kleinrock, Leonard
Korrol, C. R.
Kram, Bernard
Kurzrok, R. M.
Kuzmyak, M. G.
Lampert, Leon
Lannary, John
Liccardi, A. L.
Lindner, N. J.
London, F. H.
Low, Frank
Maedel, G. F.
Maitra, K. K.
Malina, Meyer
Marcinkowski, H. L.
Marolda, E. A.
Marshall, S. L.
Meyers, S. J.
Milenbach, Hermane
Molnar, R. J.
Ortmann, M. W.
Otis, A. N., Jr.
Paschetto, E. J.
Piccione, N. A.
Porter, R. W.
Preiss, R. J.
Rabinowich, Josef
Ragazzini, J. R.
Redmond, K. P.
Reeves, J. F.
Roberts, R. P.
Rosaler, R. C.
Rosenberg, A. E.
Sabo, J. D.
SanGiovanni, Carlo, Jr.
Sarachik, P. E.
Savage, J. E.
Sherman, Seymour
Shinners, S. M.
Slavin, M. J.
Smith, M. V.
Snyder, J. B.
Stanton, L. J.
Stone, E. S.
Svigals, Jerome
Thomas, R. O.
Trachtenberg, Alfred
Truxal, J. G.
Turchiano, M. W.
Turczyn, W. A.
Wallace, V. L.
Walton, J. S. V.
Warner, F. N.
Watkins, J. E.
Watson, E. F.
Weitman, Irwin
Wernick, J. I.
Wolfson, Richard
Wood, A. G.

Northern New Jersey Section

Aaron, M. R.
Acker, J. L.
Anderson, N. E.
Bahls, W. E.
Bearman, A. L.
Bogner, Irving
Brown, A. T., III
Brown, C. S.
Brown, R. I.
Bucher, F. X.
Bunko, Myron
Carr, G. W.
Cowles, W. W.
Davis, E. S.
Doba, Stephen, Jr.
Doniger, Jerry
Dorros, Irwin
Farber, D. J.
Garfinkel, B. D.
Glaser, W. A.
Grandmont, P. E.
Greenberg, Howard
Hamming, R. W.
Helgeson, B. P.
Huang, R. Y.
Hunter, W. B.
Kaiser, E. T.
Kelly, R. J.
Klarman, K. J.
Kulik, V. A.
Kundel, E. A., Jr.
Lazos, N. J.
Loumeau, R. F.
Lozier, J. C.
Lunney, R. E.
McCrory, J. R.
Moll, John
Mount, Ellis
Mueller, P. L.
Panter, P. F.
Podell, R. L.
Reilly, R. A.
Richards, G. P.
Ronne, J. S.
Russell, F. A.
Seckler, H. N.
Seeb, Stanley
Seergy, C. M.
Shapiro, Oscar
Sherin, David
Sippach, F. W., Jr.
Smith, E. J.

Spock, R. E.
Streeter, T. W., Jr.
Sutton, R. H.
Sweeney, W. R.
Thompson, C. F.
Turkheimer, P. M.
Wall, G. B., III
Warden, F. W.
Wilde, A. E., Jr.
Yamagami, Yonehisa
Zayac, F. R.
Zimmerman, Leonard

Princeton Section
Hellstrom, M. J.
Hoedemaker, R. W.
Lemelson, J. H.
Losher, M. I.
Rozanski, R. R. A.
Ruble, G. B.
Schofield, C. R.
Selinsky, J. J.
Surber, W. H., Jr.
Truitt, T. D.

REGION III

Atlanta Section
Coleman, B. K.
Eckel, J. R., Jr.
Glaser, H. I.
Lowman, B. L.
Pippin, R. F., Jr.
Williams, B. C.
Ziegler, N. F.

Baltimore Section
Ausfresser, H. D.
Aviles, H. N.
Axelby, G. S.
Bastow, J. G., Jr.
Behm, G. T.
Black, F. R.
Brodwin, M. E.
Buchan, J. F.
Choksy, N. H.
Cichanowicz, H. J.
Clarke, D. R.
Coppel, J. M.
Coulter, E. L.
Dietz, J. H.
Edwards, R. L., Jr.
Fegely, W. D.
Fuchs, A. M.
Gambrill, R. D.
Glaser, E. M.
Gray, A. R.
Groszer, A. J., Jr.
Hauf, J. C., III
Horn, R. E.
Hurley, W. A.
Ichniowski, F. C. J.
Jentilet, Adam
Jones, L. G. F.
Jones, W. N.
Kegel, A. G.
Kernan, Paul
Kintner, P. M.
Leahy, F. N.
Muller, J. F.
Myers, F. G.
Patton, H. W.
Penabad, Joseph, Jr.
Poston, M. H.
Randolph, G. W.
Raynes, H. D.
Roediger, F. E.
Sovill, A. W.
Spink, P. G.
Stebbins, W. J.
Stefan, R. S.
Taragin, Saul
Thomas, J. F.
Titen, Harvey
Tomlinson, C. C.
Wascavage, J. A.
Watts, H. M.
Wilson, H. C.
Wolf, H. S.
Wolpert, M. L.
Zacharia, Harry

Central Florida Section
Dibble, H. L.
Grant, W. S., Jr.
Howard, T. B.
Jahimiak, Roger

Huntsville Section
Bradley, B. C.
Cutting, Elliott
Hatcher, C. R.
Hickey, R. E.
Ingram, J. B.
Schwab, W. G.
Tu, J. C.

Miami Section
Orihuela, Adolfo
Payne, V. E.
Rosenzvaig, J. E.

North Carolina-Virginia Section
Andrews, R. E.
Bradfute, G. A., Jr.
Cockrell, W. D.
Cooper, Benjamin
Eller, J. E., Jr.
Gregory, C. A., Jr.
Imus, H. O., Jr.
Jones, T. E.
Lindeman, W. D.
Passera, A. L.
Stephens, T. L.
Young, D. B.

Northwest Florida Section
Collins, J. O.
Gamel, W. W.

Philadelphia Section
Affel, H. A., Jr.
Aires, R. H.
Anderson, W. G.
Bachofer, H. L.
Beaumariage, D. C.
Beck, Cyrus
Benner, R. H., II
Berg, N. E.
Booth, A. T., Jr.
Boyd, W. L.
Boys, H. N.
Brandt, W. E.
Brucklacher, J. E., Jr.
Bycer, B. B.
Campanella, M. J.
Caplan, D. I.
Carpenter, R. A.
Cecala, J. A.
Chronister, W. M.
Cilyo, F. F., Jr.
Cohen, B. H.
Cole, J. C., Jr.
Davis, R. W., Jr.
Dempster, B. W.
Deutsch, Joseph
Dordick, H. S.
Fabbioli, L. F.
Foster, J. A.
Friend, A. W.
Greenfield, Alexander
Gregory, T. R.
Hom, F. M.
Hoover, E. W.
Huyett, W. I.
Lee, F. F.
LeVezu, C. G.
Linden, D. A.
Linhardt, R. J.
Lovett, R. S.
Maguire, J. T.
Mathes, R. E.
McLeod, W. K.
Noonburg, W. I., Jr.
Patch, R. J.
Potosky, Maurice
Risse, J. A.
Rogers, R. F.
Rudofsky, Samuel
Seawell, W. N.
Shepard, B. R.
Shucker, Sidney
Smith, D. B.
Smith, R. V.
Sorkin, C. S.
Stephenson, J. M.
Stubbs, G. S.
Sun, Hun-Hsuan
Tou, Julius
Turner, L. C.
Tweet, B. O., Jr.
Walker, H. R.
Weiner, J. R.
Weisenberger, A. J.
Wills, W. P.
Wint, Donald
Wolin, Louis
Wolin, Samuel
Yamada, Hisao

Washington Section
Allen, D. A.
Britton, D. D.
Bryant, F. B.
Burlingame, C. W.
Bush, G. B.
Caggiano, V. J.

Calhoon, T. G.
Carruth, D. E.
Chadwell, W. L.
Chu, Yaohan
Coyle, R. J.
Dame, A. M.
Davis, M. M., Jr.
Farkas, C. E.
Finkel, Abraham
Fleming, J. J.
Gale, Morten
George, S. F.
Horowitz, Leon
Lee, A. M.
Levine, Sidney
Linvill, W. K.
Livingstone, R. H.
Looney, C. H., Jr.
Mitchell, G. J.
Morrisey, J. A.
Muller, R. M.
Notz, W. A.
O'Hara, J. J., Jr.
Ohlsson, Allan
Ostaff, W. A.
Polak, Henri
Poland, W. B., Jr.
Ramos, Edward
Rogers, A. L.
Rosenzweig, M. S.
Sanborn, G. D.
Shapiro, Gustave
Singer, J. R.
Stapleton, J. F.
Stoops, C. W.
Varela, A. A.
Waterman, Peter
White, C. F.
Wimmer, P. L.
Young, H. D.
Zastrow, K. D.

REGION IV

Akron Section
Colletti, Nicasio
Diamantides, N. D.
Flowers, H. L.
Haas, D. L.
Hann, D. D.
Lambert, C. O.
Michaelis, T. D.
Miller, J. H.
Murray, P. W.
Penniman, I. B.
Rose, W. A.
Ryburn, P. W.
Sabol, R. W.
Stahl, M. D.
Toman, W. J. V.
Yochelson, S. B.

Cincinnati Section
Baird, C. W.
Berg, D. F.
Colclaser, R. A.
Dale, W. L.
Doerr, W. H.
Engelmann, R. H.
Furukawa, M. M.
Georger, L. J.
Goff, K. W.
Hahn, R. A.
Nistico, Frank
Stewart, H. C.
Willsey, R. H.
Winkeljohann, Albert

Cleveland Section
Auth, L. V., Jr.
Craig, R. T.
Dambach, R. A.
Frenk, D. R.
Gogia, J. K.
Grasson, Walter, Jr.
Hart, C. E.
Hotchkin, E. E.
Huntley, J. R.
Kinkaid, J. C.
Klock, H. F.
Mergler, H. W.
Pfaff, R. W.
Phillips, W. E., Jr.
Post, R. H.
Saltzer, Charles
Seaton, Glenn
Tame, J. S.
Trinkle, F. J.

Columbus Section
Burgener, R. C.
Chope, H. R.
Cohen, Donald
Conlon, R. J.
Cook, E. E.
Fenwick, W. D.
McFarland, R. S.
Spergel, Philip
Weimer, F. C.
Wilson, W. A.

Dayton Section
Bornhorst, K. F.
Charbonneaux, W. A.
Finnigan, R. E.
Herrin, C. B.
Kiebert, M. V., Jr.
Lewis, D. E.
Martino, J. P.
Pastori, D. F.
Simopoulos, N. T.
Spengler, J. R.
VanWechel, R. J.
Wichmann, T. F.

Detroit Section
Barcus, Ronald
Brown, L. R.
Bublitz, A. T.
Chow, Henry
DaRoza, F. G.
El-Melehy, M. A.
Gaskill, R. A.
Genzlinger, Vance
Gilbert, E. O.
Gilbert, E. G.
Halsted, L. R.
Heckler, R. R.
Ho, Yu-Chi
Keyser, R. W.
Mathamel, F. A.
McGlinn, E. J.
Michaels, P. A.
Nakagawa, Noriyuki
Nixon, J. D.
Olson, R. G.
Rauch, L. L.
Sattinger, I. J.
Scott, D. E.
Seleno, A. A.
Sims, R. C.
Smith, Wray
Sutton, W. A.
Taplin, L. B.
Theodoroff, T. J.
Tubbs, R. J.
Webber, R. C.
Wyble, J. J.

Emporium Section
Bennett, P. E.
Haines, H. M.
Harvey, H. B.
Hoechner, I. L.
Lemley, L. W.
Oblinger, J. T.
Pryima, R. M.
Seeley, R. M., Jr.

Pittsburgh Section
Barnard, J. D.
Brendle, T. A.
Caywood, W. P., Jr.
Ellison, B. P.
Fulmer, L. C.
Gocsik, J. M.
Halprin, L. H.
Kinder, H. R.
Knowles, C. R.
O'Donnell, J. J.
Rau, F. J.
Rogers, L. J.
Royer, G. H.
Starbuck, W. H.
Strull, Gene
Sze, T. W.
Werst, M. C.

Toledo Section
Ewing, D. J., Jr.
Spademan, C. F.

Williamsport Section
Webb, H. E.

REGION V

Cedar Rapids Section
Hedgcock, W. T., Jr.
Lowenberg, E. C.

Chicago Section
Antonelli, D. R.
Axelrod, L. R.
Bergen, H. A.
Bold, N. T.
Buchta, J. C.

Bullen, C. V.
Bymberg, R. J., Jr.
Carter, Robert
Cermak, C. W.
Chang, Bansun
Chulsky, Isadore
Cooney, J. J.
Dolce, S. L.
Druz, W. S.
Dunbar, E. A.
Epley, D. L.
Ferre, G. E.
Fu, King-Sun
Gerlach, A. B.
Greenberg, C. J.
Gregory, E. C.
Hansen, A. G., Jr.
Hoffman, C. H.
Hori, Shizuo
Isolampi, G. E.
Jenness, R. R.
Kott, W. O.
Kreer, J. B.
Kuhn, N. J.
Lafferty, V. C.
Lee, D. K. K.
Leth, T. R.
Lewis, H. A.
Li, Ching-Chung
Martin, J. W., Jr.
Merz, R. A.
Meyer, Andrew
Mitchell, F. R.
Mittelmann, Eugene
Nauer, M. J.
Noble, D. S.
O'Neill, R. M.
Saltzberg, Theodore
Shewan, William
Shively, R. R.
Smithana, D. R.
Stan, John
Thielen, L. R.
Vaicunas, A. A.
VanBosse, J. G.
VanNess, J. E.
VanValkenburg, M. E.
Verbanec, W. R.
Warshawsky, Jay
Weissert, R. K.

Evansville-Owensboro Section
Kercher, D. L.

Fort Wayne Section
Ackworth, D. L.
Brady, F. H.
Emery, R. C.
Kalish, J. H.
Kaplan, Robert
Norris, B. J.
Richeson, W. E.
Solomon, R. M.

Indianapolis Section
Billheimer, A. C.
Evans, R. A.
Garofalo, A. D.
Grant, M. P.
Hammond, S. B.
Longren, W. K.
McCrocklin, R. E.
Ogborn, L. L.
Pearson, S. J.
Smith, C. C. L.

Louisville Section
Kwo, T. T.
Means, T. S., Jr.

Milwaukee Section
Arakelian, G. P.
Babladelis, George
Cork, H. A.
Gessner, Urs
Goodman, P. H.
Graham, J. D.
Haraldsen, H. P.
Jensen, K. S.
Lind, E. R.
Makela, L. V.
Mezger, J. P.
Min, H. S.
Morin, D. C., Jr.
Pierce, R. L.
Rekoff, M. G., Jr.
Schlager, K. J.
Schwartz, E. B.
Smith, C. C.
Zelazo, N. K.

Twin Cities Section
Adams, G. E.
Adkisson, W. M.
Alderson, R. C.
Alfsen, G. F.
Allen, D. H.
Anderson, L. T.
Balzart, E. J., Jr.
Bartlett, V. W.
Beaudoin, P. E.
Benassi, D. A.
Bergan, K. N.
Bock, E. D.
Brunetti, Cledo
Carlson, R. A.
Clark, R. N.
Croze, M. W.
Cummings, K. C.
Dundovic, J. F.
Dunwell, R. D.
Fox, A. J.
Gilson, J. R.
Gise, F. G., Jr.
Gustafson, H. A.
Hardenbergh, G. A.
Hird, F. S.
Hulstrand, B. E.
Inman, T. F.
Johnson, H. W.
Kershaw, J. A.
Ketchum, J. R.
Koenig, J. D.
Lahue, P. M.
Lanzkron, R. W.
Lindemann, A. W.
Lode, Tenny
Ludwig, J. T.
Macomber, G. R.
Markusen, D. L.
Maze, R. O.
McLane, R. C.
Moe, W. J.
Muckenhirn, O. W.
Murphy, G. J.
Nellis, W. M.
Peatman, J. B.
Rowland, C. A., Jr.
Schuck, O. H.
Sear, A. W.
Senstad, P. D.
Stewart, W. E.
Stone, N. T.
Swanlund, G. D.
Toth, D. H.
Vogel, J. P.

REGION VI

Alamogordo-Holloman Section
Bauman, E. J.
Kuerschner, Helmut
Zimmerman, A. P.

Dallas Section
Askew, W. J., Jr.
Braun, Clarence
Buehrle, C. D.
Buzard, R. S.
Creager, L. D.
Harmon, F. I.
Heizer, K. W.
Johnson, G. D.
McDonald, Marshall
Miller, N. D.
Morris, B. V.
Ocnaschek, F. J.
Pittman, P. D., Jr.
Prier, H. W.
Stanton, A. N.
Tatum, F. W.
Wadel, L. B.

Denver Section
Daniels, W. H.
Finch, M. D.
Messler, F. J.
Mielziner, Walter
Morris, W. L.
Morroni, D. J.
Ostwald, L. T.
Schneebeck, D. A.
Tary, J. J.

El Paso Section
Emerling, R. A.
Rojas H, A. M.

Fort Worth Section
Beckman, W. R.
Blasingame, J. J.
Cone, J. H.
Evans, W. L.
Heizer, L. E.
Jiles, C. W.
Lowrie, G. M.
Teasdale, A. R., Jr.

Watkins, O. E.
Young, F. W.

Houston Section
Bucy, J. F., Jr.
Easterling, M. F.
Francis, L. G.
Frobese, C. W.
Gentry, J. A.
Hutchens, R. L.
Navarro, S. O.
Tasini, Betsalel
Waldie, A. D.

Kansas City Section
Breyfogle, L. D., III
Hickey, L. F.
Miller, H. G.
Murray, W. A.
Stout, H. L.
Wilcox, J. V.

Little Rock Section
Cannon, W. W.

Lubbock Section
Estes, S. E.
Perkins, C. S.
Tomlinson, Z. G.

New Orleans Section
Drake, R. L.

Oklahoma City Section
Grubbs, C. E.
Krystek, M. E.
Ledbetter, R. P.
Puckett, T. H.
Silva, R. F.
Vlay, G. J.

St. Louis Section
Allison, W. H.
Arndt, R. L.
Hibbits, R. M.
Lago, G. V.
Malsbary, J. S.
Mayer, M. F., Jr.
Mohrman, R. F.
Mutchek, J. H.
Norman, C. F.
Reed, D. L.
Salman, N. D.
Sayer, J. D.
Tedeschi, Anthony
Twombly, J. W., Jr.
Weinstock, G. L., Jr.
Winter, D. F.

San Antonio Section
Bostick, F. X.
DuBose, G. P., Jr.
Hirsch, C. O.
Hoffman, A. A. J.
Mayleben, E. F.
Reinhard, E. A.
Ziemer, D. R.

Tulsa Section
Brashear, R. T.
Cairns, T. W.
Day, C. E.
Fox, D. N.
Labarthe, L. C.
Laird, J. A., III
O'Brien, D. G.
Rowley, R. G., Jr.
Silverman, Daniel
Sykora, G. E.

REGION VII

Albuquerque-Los Alamos Section
Ehrman, Leonard
Katzenstein, Jack
Pace, T. L.
Shephard, R. W.

China Lake Section
Crawford, J. A.
Creusere, M. C.
Kim, P. K. S.
Schimmel, George

Hawaii Section
Jones, R. C., Jr.

Los Angeles Section
Abbott, W. R.
Ackerlind, Erik
Akin, P. A.
Albrecht, Albert
Anderson, Frank
Andrews, L. A.
Anzel, B. M.
Aroyan, G. F.
Aseltine, J. A.
Avrech, Norman
Baker, D. L.
Ballard, K. C.
Barlett, F. R.
Barnes, J. L.
Bayley, L. B., Jr.
Beckwith, H. W.
Beecher, A. E.
Bekey, G. A.
Bement, W. A.
Bible, R. E.
Bills, G. W.
Bonney, R. B.
Borgeson, P. W.
Bower, J. L.
Braverman, D. J.
Broadwell, W. B.
Brock, P. A.
Brown, D. E.
Buchman, W. W.
Buland, R. N.
Burk, W. A.
Burnsweig, Joseph, Jr.
Carlson, A. R.
Carlson, C. O.
Carlson, R. S.
Cassidy, R. E.
Chandaket, Prapat
Chandler, D. P.
Christensen, A. V.
Cordray, R. E.
Corvi, J. A.
Cosgrave, S. J.
Curry, W. S., Jr.
Deaux, F. J.
Deming, A. F.
Deuser, D. A.
Dickinson, H. B.
Diem, C. W.
Diemer, F. P.
Dinning, J. R.
Doty, R. L.
Drucker, Alvin
Dzilvelis, A. A.
Edelsohn, C. R.
Eikelman, J. A., Jr.
Engel, H. L.
Eno, R. F.
Fernandey, Ferdinand
Finley, W. A.
Fish, W. Y.
Forbath, F. P.
Foxman, Eugene
Francis, T. F.
Frankel, Sidney
Frankos, D. T.
Fuller, R. H.
Fulton, A. S.
Furumoto, Nobuo
Gabler, R. T.
Gaitan, T. C.
Garber, L. F.
Gauronskas, P. P.
Gaylord, R. S.
Gerardi, F. R.
Gerken, G. H.
Gill, W. J.
Ginstling, Ajzyk
Grabbe, E. M.
Graham, J. D.
Gross, William
Gunning, W. F.
Hadden, F. A.
Hansen, O. B.
Harmon, W. G.
Harrington, L. M.
Hassel, R. R.
Hayes, J. E.
Heffner, E. K.
Heyliger, G. E.
Hicks, A. R.
Hruby, R. J.
Hughes, Frank
Hutcheon, R. S.
Izuel, A. G.
Jack, R. W.
Jackson, K. R.
Jacobs, J. E.
Janeway, R. K.
Joerger, J. C.
Johnson, J. J.
Johnson, R. W.
Johnson, W. A.
Juran, Warren
Kaufman, F. H.
Kaufman, Sidney
Kawahata, B. I.
Kennel, J. M.
Keppel, R. A.
Kerster, George
King, C. G., Jr.

King, J. E.
Kirsch, H. A.
Kishi, F. H.
Klein, M. L.
Knox, R. V.
Krames, C. V.
Krill, C. K.
Kroy, W. H., Jr.
Lawrence, A. F., III
Lee, H. J.
Leondes, C. T.
Leone, W. C.
Levinson, R. M.
Lewis, D. E.
Liang, Ming-Tsu, M.
Lillibridge, E. H.
Lords, F. V.
Louie, William
Lyons, L. H.
Malone, Martin
Mancini, A. R.
Mankinen, E. J.
Manly, Ron
Margolis, Maier
Mayberry, L. A.
McRuer, D. T.
Mehner, E. W.
Mileson, D. F.
Miller, D. S.
Mitsutmoi, Takashi
Morrison, A. I.
Morton, W. B., Jr.
Myers, W. A.
Nelson, C. S., Jr.
Neumann, Leopold
Noland, A. R.
Nuban, Ebrahim
Nuttall, H. V.
O'Brien, W. C.
Olsen, L. V.
Parker, A. T.
Pernick, L. J.
Post, Geoffrey
Poulson, W. A.
Primozich, F. G.
Putter, Klaus
Quackenbush, R. E.
Radant, M. E.
Raffensperger, M. J.
Ramer, F. H., Jr.
Ramstedt, C. F.
Redden, E. T.
Redmond, J. G.
Rehler, K. M.
Rescoe, J. M.
Rickords, T. J.
Rifkind, Jesse
Robertson, G. R.
Rogers, J. G.
Rogers, T. A.
Romano, A. J.
Rosenstein, A. B.
Rosenthal, G. W.
Rowe, D. E.
Ruiz, M. L.
Salzer, J. M.
Samuels, A. H.
Sanneman, R. W.
Sarture, C. W.

Savant, C. J., Jr.
Savo, T. A.
Sawyer, H. F.
Scammel, B. C.
Schalk, Norbert
Schroeder, William
Schultz, P. R.
Schultz, R. T.
Schulz, K. S.
Scott, W. F.
Sensiper, Samuel
Shelley, R. G.
Shenk, J. W.
Shimada, George
Short, F. E.
Shuler, M. H.
Shultise, Q. M.
Shutt, S. G.
Siegel, J. C.
Silva, L. M.
Sink, R. L.
Slocomb, G. M.
Smith, J. C.
Smith, J. E.
Snapp, K. M.
Snyder, W. A.
Sohler, J. F.
Stark, H. P.
Steinkolk, R. B.
Stimpson, L. D., Jr.
Stout, T. M.
Sturm, T. F.
Takahashi, Kiyoshi
Thompson, D. M.
Thorensen, Ragnar
Thorpe, L. M.
Turn, Rein
Valery, N. A.
VanCuren, Verlyn
Vega, C. J.
Vittum, W. M.
Vodovoz, Erwin
Vulliet, P. O.
Wachowski, H. M.
Waddell, R. L.
Wakamiya, Yooichi
Walker, N. L.
Walkup, L. A.
Wallace, Charles, Jr.
Walp, R. M.
Walters, L. G.
Wanlass, S. D.
Warrington, Wilmer, Jr.
Watkins, E. L.
Wedel, J. J., Jr.
Wennerberg, Gunnar
Wenters, R. L.
White, L. M.
Whitford, R. K.
Wolman, L. L.
Wong, D. S.
Wong, E. C.
Wong, Herbert
Young, W. L.
Zabusky, N. J.
Zacharias, Robert
Ziegler, R. M.
Zimmerman, R. L.
Zoller, C. J.

Phoenix Section

Ballantine, J. H.
Levine, Daniel
Ross, J. M.
Scrafford, R. L.
Sutton, J. G., Jr.

Portland Section

Bowman, D. G.
Doel, Dean
Jenkins, R. W.
Mendoza, D. C.
Stone, L. N.

Salt Lake City Section

Clegg, J. C.
Gilmour, G. A.
Jackson, H. L.
Murphy, L. C.

San Diego Section

Biering, A. H.
Campbell, Robert
Cox, T. M.
Dodd, G. M.
Ferner, R. O.
Flarity, W. H.
Fogel, L. J.
Greenspan, L. E.
Kalbfell, D. C.
Klimberg, Joseph
Lay, P. D.
Mealey, G. J.
Okahata, E. S.
Shechet, M. L.
Wade, Ernest

San Francisco Section

Alexander, C. S., Jr.
Berryhill, J. L.
Bharucha, B. H.
Binnall, E. P.
Brennan, R. D.
Buntenbach, R. W.
Chesebro, E. L.
Chin, B. C.
Davy, L. H.
Dibb, George
Durfey, G. K.
Enos, R. M.
Farman-Farmaian, Ghaffar
Firschein, Oscar
Gardiner, K. W.
Gentry, E. B.
Gleason, C. A.
Goodall, J. R., Jr.
Gyllstrom, N. D.
Hallman, A. B.
Hexem, John
Iwama, Morimi
Jameson, R. J.
Jury, E. I.
Kazanjian, H. A.
Kerwin, W. J.
Kincas, J. W.
Klotter, Karl
Kochenderfer, W. E., Jr.
Kortman, C. M.
Landsman, Louis

Lendaris, G. G.
Lessley, T. D.
Lindberg, H. E.
Martinez, H. M.
Mullin, F. J.
Ness, R. J.
Nishizaki, Ray
Oliver, B. M.
Over, J. J., Jr.
Pope, J. C.
Pringle, Ralph, Jr.
Rearick, H. F.
Regenos, K. M.
Ross, Albert
Samario, E. J.
Smith, D. L.
Smith, O. J. M.
Storke, F. P., Jr.
Tudor, B. J.
Tuttle, D. F., Jr.
Vea, T. H.
Wang, P. K. C.
Watson, J. K.
Windsor, R. N.

Seattle Section
Biggs, J. D., Jr.
Birch, J. S.
Bishop, D. J.
Galloway, W. C.
Homitch, J. M.
Ledray, William
Miller, J. J., Jr.
Noland, L. J.
Smith, R. A.
Stapleton, E. R.
Vermilion, E. E.
Whipple, M. M.

Tucson Section
Bard, W. E.
Crow, R. B.
Lindenberg, E. C.
Martin, L. C.
Sakrison, D. J.
Simpson, E. J.

REGION VIII

Bay of Quinte Section
Brule, R. J.
Flemons, R. S.
MacKelvie, J. S.

Hamilton Section
Carnahan, C. W.
Kassner, John
Rogers, A. E.

London Section
Fletcher, H. R.
Stroud, E. L.

Montreal Section
Bar-Urian, Moshe
Birman, Gerhard
Caron, J. Y.
Cummins, J. A.
Demers, Pierre
Germain, L. V.
Ginsburg, Isaac
Heckman, G. R.
Malcolm, F. W.
Prichodjko, Alexander
Reeves, Rene
Richard, G. B.
Rinfret, C. J.
St. Onge, J. L.
Semple, E. R.
Sproul, R. W.

Northern Alberta Section
Wood, D. R.

Ottawa Section
Beneteau, P. J.
Newton, K. G. D.

Toronto Section
Baldwin, J. H.
Balmain, K. G.
Beaudoin, D. P.
Byers, H. G.
Carew, S. J. H.
Carley, R. R.
Elliott, W. F.
Hackbusch, R. A.
Lang, G. R.
Maw, J. C.
McCloskey, K. P.
Newhall, E. E.
Otsuki, J. S.
Penrose, R. M.
Peprnik, H. O.
Stoddart, T. W. H.
Unger, J. H. W.
Wall, Ernest

Vancouver Section
Bohn, E. V.
Hewit, H. O.
Kersey, L. R.
McDonald, W. H.
Moore, A. D.

OVERSEAS MILITARY
Cassidy, J. J.
Dunham, P. N.

FOREIGN SECTIONS

Buenos Aires Section
Pinasco, S. F.

Israel Section
Shamir, Jedidiah
Weislitzer, Josef

Rio de Janeiro Section
DeMattos, H. C.

Tokyo Section
Honda, Tatsuo
Ibuka, Masaru
Ishikawa, Takeji
Iwakata, Hideo
Koichibara, Tadashi
Konomi, Mitsugu
Mita, Shigeru
Morita, Masasuke
Nakahara, Fujio
Nishino, Osamu
Okada, Minoru
Owaki, Kenichi
Tanabe, Yoshitoshi
Tanaka, Yoneji
Taniguchi, Fusao
Yano, Akira

FOREIGN

Australia
Brodribb, M. I.
Davies, R. J. C.
Honnor, W. W.

Brazil
Barros-Barreto, L. A. G. C.
Waeny, J. C. C.

Denmark
Jarlov, A. L.

England
Clark, J. E.
Cullen, A. L.
Dawes, E. J.
Dietiker, Walter
Fleming-Williams, B. C.
Harris, K. E.
Laverick, Elizabeth
Parsons, A. N.
Warr, H. J. J.

France
Baron, Jean
Berline, S. D.
Blachier, B. L.
Ferrier, P. A.
Ghertman, Jean
Girerd, J. L. M.
Loeb, J. M.

Germany
Busch, C. W.
Peters, J. F.
Rohde, Lothar
Uhrmann, A. M.

Holland
Alma, G. H. P.
Janssen, J. M. L.
Tellegen, B. D. H.

India
Mirchandani, I. T.
Mishra, Srinibas
Mukerji, M. M.

Italy
Bacchialoni, F. L.
DeDominicis, C. M.
Egidi, Caludio
Pinolini, F.
Tchou, Mario
Verdoni, L. G.
Vergani, Angelo

Lebanon
Hoffman, J. D.

Mexico
Hernandez-Ramos, A. M.
Suarez-Diaz, Jorge

Norway
Engvik, S. B.

Puerto Rico
Maldonado, Bolivar

Sweden
Andersson, K. N.
Ekelof, Stig
Elfving, A. L.
Fagerlind, S. G.
Gyllenkrok, Thure-Gabriel
Josephson, B. A. S.
Lofgren, E. O.
Roll, Anders
Romell, G. D. R.
Svala, C. G.

Switzerland
Braun, A. F.
Demieville, Henri
Shah, R. R.
Strohschneider, Walter
Thalmann, Victor
Weber, G. C.

Venezuela
Arreaza, R. G.
Kinzbruner, Paul